Networks & Algorithms

for
High School Students

developed by the
participants of the
Institute of Networks & Algorithms
Rutgers University
New Brunswick, New Jersey
July, 1989

edited by
Barbara McIlroy
Patricia Van Hise
July, 1990

INTRODUCTION

The materials in this booklet were developed by the participants in the 1989 summer institute at Rutgers University in discrete mathematics, entitled "Networks and Algorithms," and were tested in the classroom during the 1989-90 school year. They were compiled and edited during the 1990 Institute.

Names of the participants are listed in Appendix A. Names of those who contributed to this booklet as well as any Institute resource used, are at the bottom of each page in square brackets; resources are listed with capital letters, contributers in lower case. Two resources that have been extremely useful, and are highly recommended for teachers, are Copes, et. al and Cozzens.

The booklet is divided into four major sections. Each begins with an outline, which suggests where the materials might be used. Some of the materials are for the teachers; some are for the students. Teaching notes include enough information for the teacher with no prior knowledge of discrete mathematics to be able to teach this material. A glossary of terms is included in Appendix B and more information can be found in the references listed in Appendix C.

The topics in this booklet are appropriate for high school students in all levels. Some topics use basic mathematical computation and work well in general math classes. Others involve algebra, geometry, and analytic geometry skills and are appropriate for these classes as well as for calculus and for classes explicitly in discrete mathematics.

We wish to express our thanks to those 1989 Institute Participants who were present at the 1990 Leadership In Discrete Mathematics Institute and always were ready to confer with us on the preparation of this booklet:

David Farber
Joan Shyers
Susan Simon
Philip Reynolds
Giselle Zangari

Special thanks to the Director of the Institutes, Joseph G. Rosenstein, for his inspiration and guidance.

PREFACE

This is a booklet!

That's what the introduction says. But, as you can plainly see, it's a bit longer than what the term booklet is usually applied to. It started out small, but it plainly grew!

The "booklet" represents a great deal of effort by Barbara McIlroy and Pat VanHise, who organized all the materials submitted by the participants of the summer 1989 program, and by Linda Holt, who subsequently typed and refined the manuscript. Barbara and Pat were participants in the 1989 program, and Linda was a graduate student in mathematics who served as a graduate assistant during the 1990-1991 academic year. All of us are indebted to them.

What this booklet shows is that topics in discrete mathematics can be adapted quite readily for high school and middle school students. These materials were in fact all used by teachers in their own classrooms.

These materials are not really stand-alone materials -- that is, they often require background information that is not presented here. They are also not necessarily original, but attempts have been made to credit the sources that were used. They are intended to be used as a resource for teachers, particularly those in the Rutgers Leadership Program in Discrete Mathematics, who have been exposed to the topics discussed here.

We hope that those teachers who use these materials will submit their suggestions and revisions for inclusion in future versions of the booklet. After several iterations, the "booklet" may turn into a "book" and be distributed more widely.

Discrete mathematics provides us with unique opportunities to implement the NCTM Standards in our classrooms and schools. Adopting student oriented instructional techniques, emphasizing discovery and understanding, providing open-ended problems which are close to the frontiers of knowledge, dealing with real world problems -- all are easier to do using discrete mathematics topics.

I hope that you will find this booklet useful with your students and your colleagues!

Communications about this booklet should be sent to CMSCE -- Leadership Program, P.O.Box 10867, New Brunswick, New Jersey 08906. I can be reached by email at joer@math.rutgers.edu or by phone at 908/932-4065.

Joseph G. Rosenstein
Professor of Mathematics
Director, Leadership Program in Discrete Mathematics

March 1991

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(NOTE: Each chapter has its own Table of Contents.)

GRAPH THEORY CONTENTS

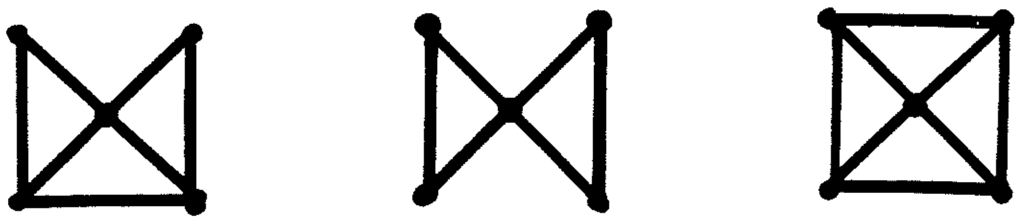
TEACHING NOTES

NOTE 1: ***Three Themes of Discrete Mathematics***

1. *Existence - Is there a solution?*
2. *Counting - How many solutions are there?*
3. *Efficiency - What is the best solution?*

Let's look at how these three questions are addressed on a problem dealing with *__graphs.__* *A graph is defined as a set of vertices (points) and edges (connecting line segments).*

__Example 1__ *__Example 2__* *__Example 3__*

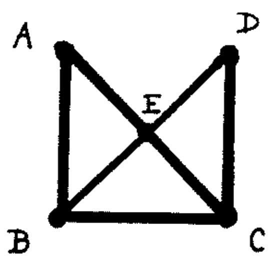


Which of these graphs can you trace by passing over each edge exactly once and not picking up your pencil? In other words, does there exist an Eulerian path? Does there exist an Eulerian circuit? (The path is called an Eulerian circuit if the starting and ending vertices are the same.) Why or why not? If so, is there more than one path? How many? Is any path shorter than the others? Longer?

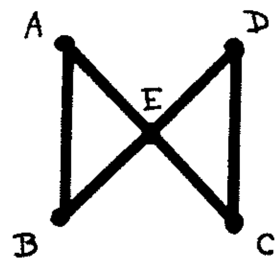
[Van Hise]

SOLUTIONS:

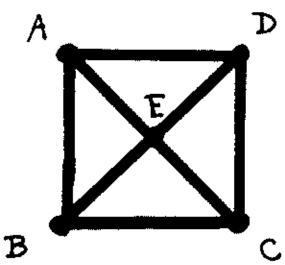
Example 1



Example 2



Example 3



*Example 1 has 4 Eulerian paths starting at vertex B and ending at vertex C and 4 Eulerian paths starting at C and ending at B. (*** What conjecture can be made? ***)*

BAEDCEBC	*BEDCEABC*	*BEABCEDC*	*BCEABEDC*
CBEBDEAB	*CBAECDEB*	*CDECBAEB*	*CDEBAECB*

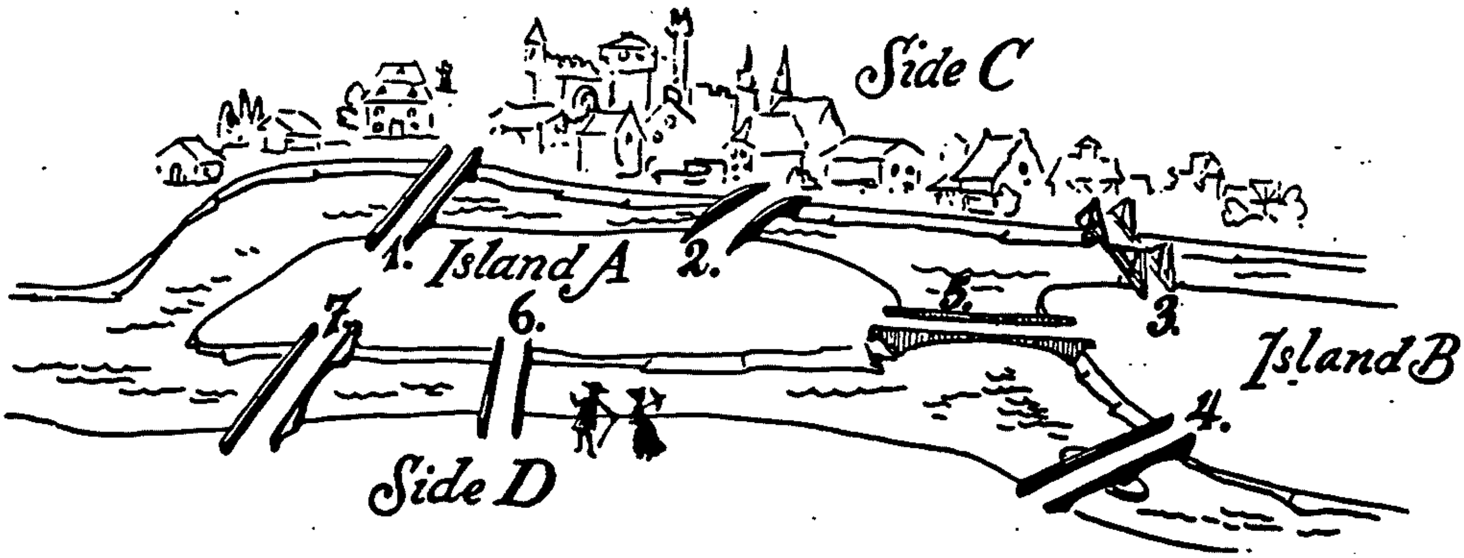
*Example 2 has 4 Eulerian paths starting and ending at vertex A (all Eulerian circuits), 4 at B, 4 at C and 4 at D. (*** What do all these vertices have in common? ***)*

AECDEBA	*AEDCEBA*	*ABEDCEA*	*ABECDEA*
BECDEAB	*BEDCEAB*	*BAEDCEB*	*BAECDEB*
CEBAEDC	*CEABEDC*	*CDEBAEC*	*CDEABEC*
DEABECD	*DEABECD*	*DCEBAED*	*DCEABED*

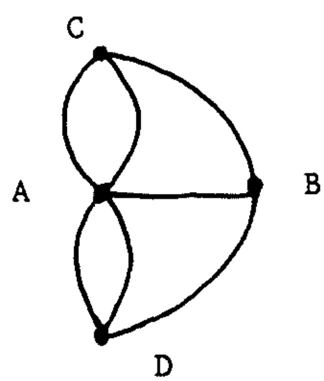
Example 3 has ***NO*** *Eulerian path!! Why? You may have noticed that in Example 1, two vertices, B and C, have an odd number of edges; the other vertices in Example 1, and all the vertices in Example 2, have an even number of edges. In Example 3, however, there are four odd vertices. Any Eulerian path must leave each vertex as often as it arrives, so that if the path includes each edge, every vertex - except the beginning and the end of the path -must be even. So at most two vertices may be odd. But Example 3 has more than 2 odd vertices. If there are 0 odd vertices many paths exist, each starting and stopping at the same vertex. If there are 2 odd vertices, as in Example 1, paths exist which must start at one of the odd vertices and end at the other.*

*NOTE 2: **WHERE DID GRAPH THEORY START? The Seven Bridges of Koenigsberg***

The town of Koenigsberg in old Germany was famous for its seven bridges. Six of these bridges ran from the two banks of the Pregel River to each of the two islands shown. The seventh bridge connected the two islands. The story tells us that the mayor of the town offered the hand of his daughter in marriage to anyone who could walk across each of the seven bridges without recrossing any of the bridges? Can you do it? Try tracing a path without lifting your pencil from the paper or recrossing any bridges.



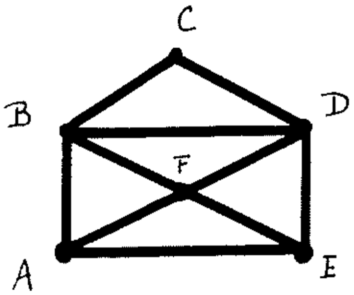
The construction of a graph as a model of this situation is helpful to study the problem.



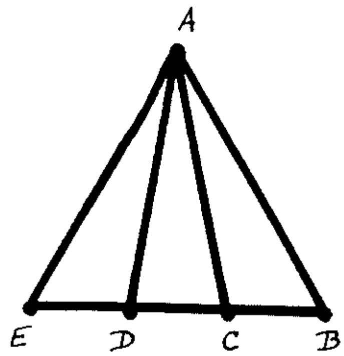
The vertices represent the land masses and the edges represent the bridges. Leonhard Euler solved this problem in 1736 by showing that to cross each bridge exactly once and return to the starting point, each vertex must be the meeting point of an even number of edges, called the degree of the vertex. Since each vertex is odd in the above graph, it cannot be done in this case.

NOTE 3: ***Euler Paths and Euler Circuits***

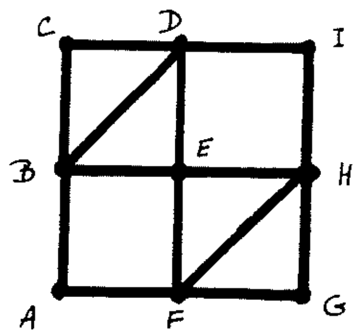
A graph consists of edges (line segments) and vertices (the endpoints of the line segments). The figures below are all examples of graphs. Which of the following graphs can be drawn without removing the pencil from the paper and without retracing any lines?



Graph I

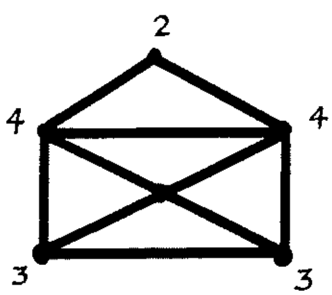


Graph II

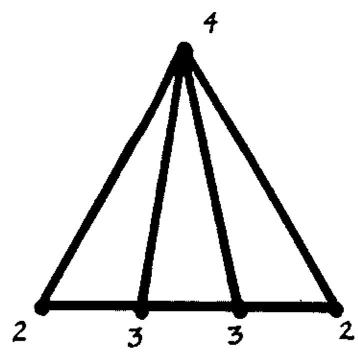


Graph III

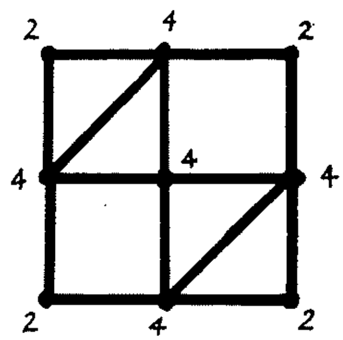
Find the degree of each of the vertices of Graph I, II, and III. The degrees are entered next to each vertex on the graphs below.



Graph I



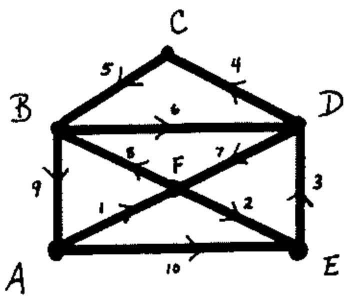
Graph II



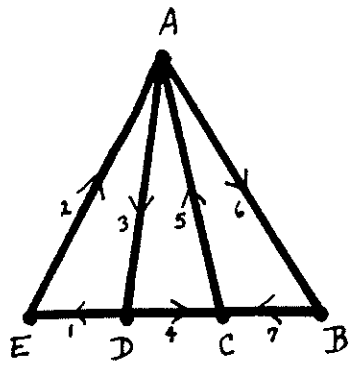
Graph III

Notice that Graphs I and II have two odd vertices each. Graph III is the only graph where all the vertices are of even degree. Graph III can be drawn without retracing any edges and the path ends at the beginning vertex! A path that uses each edge in a graph exactly once and ends at a vertex different from the starting vertex is called an *Eulerian path.* *A path that starts and ends at the same same vertex is a circuit. A path that uses each edge in a graph exactly once and ends at the starting vertex is called an* *Eulerian circuit.* *To have an Eulerian path or circuit, the graph must be connected. A graph is connected if each pair of vertices can be joined by a path. A graph has an Eulerian circuit if the graph is connected and each vertex has an even degree. The circuit can start at any vertex. A graph has an Eulerian path if the graph is connected and exactly two of the vertices have odd degrees. The path can start at either of the odd vertices and ends at the other odd vertex.*

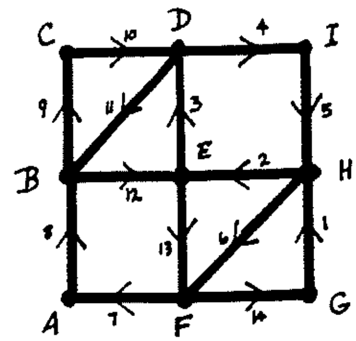
[COZZENS,Reynolds,Zangari]



Graph I



Graph II



Graph III

Think of Graphs I,II, and III as rural postal routes with all the mailboxes on one side of the road. Tracing out the graph without removing the pencil from the paper and without retracing any lines gives a route for the postman to deliver the mail with the least amount of driving since no edge is retraced. The key therefore is to start at any vertex and trace out the graph without retracing any of its edges.

USING THE DATA GATHERED FROM THE LAB, ANSWER THE FOLLOWING QUESTIONS.

1. For all the graphs you said were traceable, how many odd vertices existed in each graph?

2. For all the graphs you said were not traceable, how many odd vertices existed in each graph?

3. Using your answers to questions [1] and [2], can you write a test to determine if an Eulerian path exists for a given graph?

4. Can a graph have an Eulerian path if it is not connected? Be sure you have included your answer to this question in your test in problem 3.

5. If a graph has odd vertices, is it possible for the graph to have an odd number of odd vertices?

6. What is a test to determine if there is an Eulerian path which will end where it started?

7. What is a test to determine if there is an Eulerian path which ends in a different vertex than where it started?

8. Given any graph. A new graph is made by removing exactly one edge. What is true of the change in the number of odd vertices when comparing the new graph to the original graph?

9. Draw a graph with exactly one odd vertex.

ANSWERS TO EULER ACTIVITY #1

1. At most two

2. More that two odd vertices

3. The graph must be a connected graph and have at most two odd vertices.

4. No

5. No. Each edge has two endpoints, so the total number of endpoints is time the number of edges, an even number. If there were an odd number of add vertices, they would account altogether for an odd number of endpoints; the even vertices would account altogether for an even number of endpoints, so the total number of endpoints would be odd -- which can't happen.

6. All vertices must be even. Note: The Eulerian path can start at any of the vertices.

7. There must be exactly two odd vertices. Note: The Eulerian path must begin at one of the odd vertices and end at the other odd vertex.

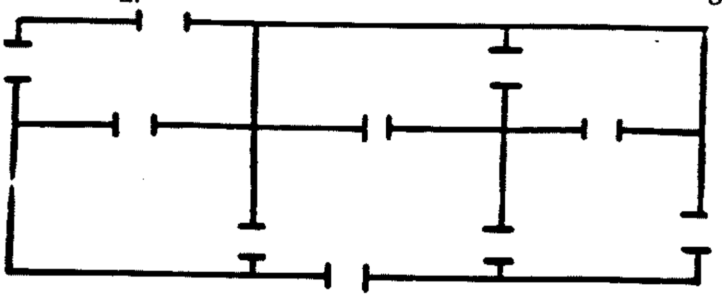
8. Remove an edge between two odd vertices -- number of odd edges decreases by two.
Remove an edge between an odd and an even vertex -- number of odd edges remains the same.
Remove an edge between two even vertices -- number of odd edges increases by two.

9. Not possible. See the answer to 5.

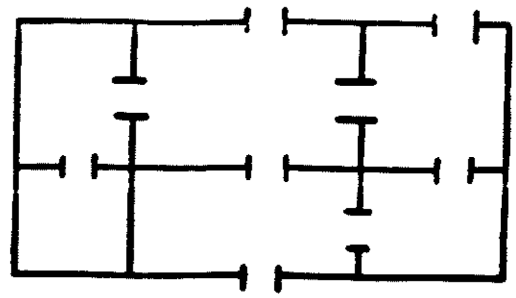
EULER ACTIVITY #2: Applications of Eulerian paths

1. The figures below are the floor plans of houses. Decide if it is possible to walk through every doorway once and only once. You may start anywhere. Draw an Euler path, if one exists.

a.

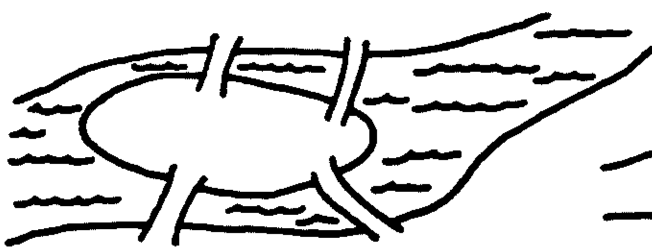


b.

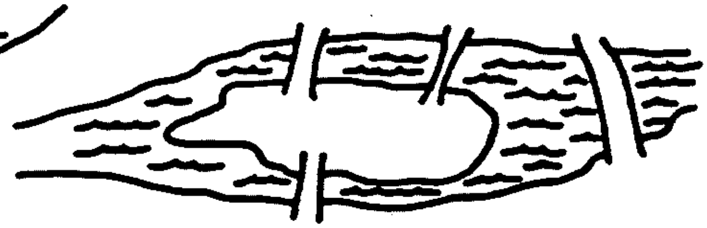


2. The drawings below are pictures of rivers with bridges and islands. Draw a graph corresponding to each picture and decide if it is possible to walk across each bridge exactly once and return to the starting point.

a.

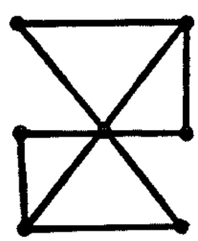


b.

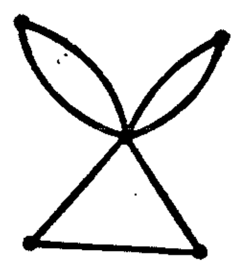


3. For each of the graphs below, decide if the figure can be drawn without removing your pencil from the paper and without retracing any lines.

a.



b.

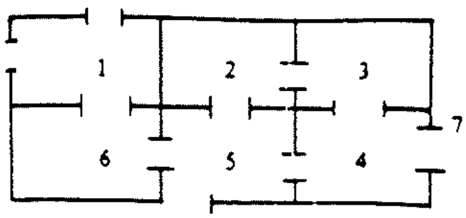


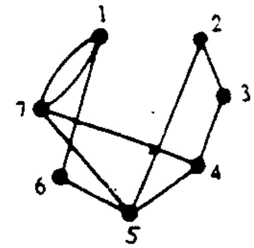
4. Create an algorithm that allows you to draw an Euler path for a graph, if one exists.

[COZZENS/Johnson]

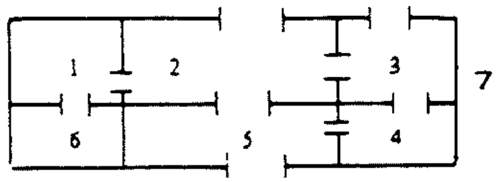
SOLUTIONS:

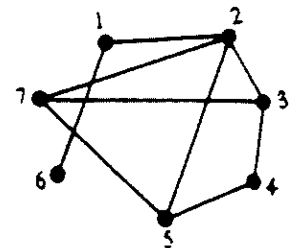
1. a. Yes, if you start in room 1 and end in room 3, or vice versa.



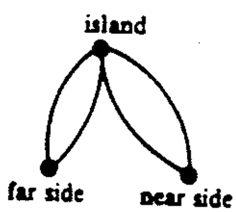


b. No, vertices 3, 5, 6, 7 have an odd number of edges incident to them.

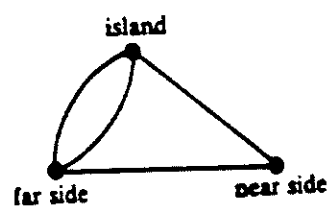




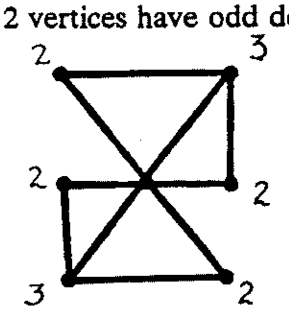
2. a. yes

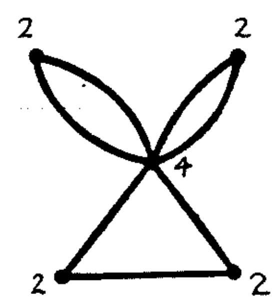


b. no



3. a. Yes, exactly 2 vertices have odd degree. b. Yes, all vertices have even degree.





4. One solution:

1. If the vertices are all even:
   (a) Choose a vertex to start the path. Label this vertex S for start. The path will also end at 5.
   (b) Trace a circuit that begins and ends at S.
2. If there are exactly two odd vertices:
   (a) Choose one of the two odd vertices to start the path. Label this vertex S for start.
   (b) Choose the second odd vertex to end the path. Label this vertex E for end.
   (c) Trace a circuit that begins at S and ends at E.

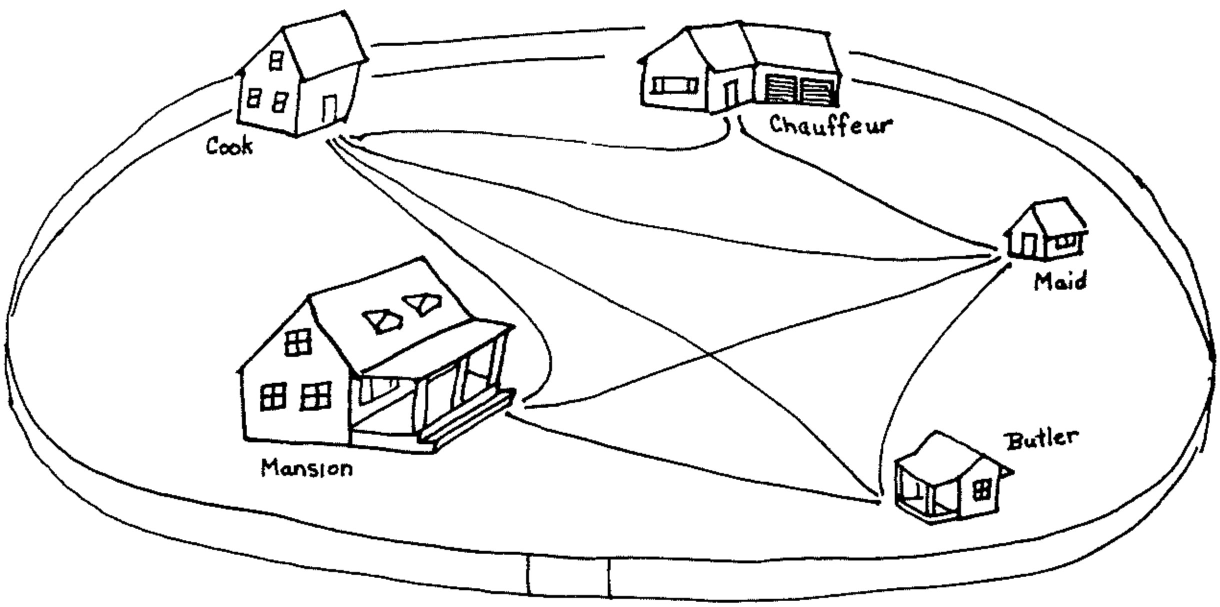
EULER ACTIVITY #3: "The Case of the Stolen Diamonds"

Teacher Note:

The Case of the Stolen Diamonds should be acted by students using as many props as possible. After the presentation and time to guess, the class should use a graph to solve the mystery. The houses can be the vertices and the paths to each house the edges. The mansion (the original vertex) is odd and the only other odd vertex is the butler's house. Therefore, the point of termination in this Eulerian path is the butler's house.

Similarly, Goldini's trick, acted by students is a powerful activity to reinforce the Eulerian Path Theorem.

THE CASE OF THE STOLEN DIAMONDS



It was a snowy night when the family jewels were stolen from the Shmendrick Mansion. As soon as the alarm sounded, the gates around the estate were locked. The jewels were somewhere on the grounds. But where? Were they in the cook's quarters, or perhaps the chauffeur's cottage? There was only one person who could crack this case: world famous inspector Euler Toots of Scotland Yard. Early the next morning Inspector Toots came to visit the estate in search of clues.

[GOLDSTEIN/Reynolds/Zangari]

This is what Inspector Toots found: fresh tracks crisscrossing the grounds. Toots couldn't tell which direction they were going, but was sure that each track had only been used once. The Inspector was careful to make a map of the tracks before the snow melted.

At noon, all of the principals in the case were brought into the main dining hall, just as Inspector Toots was hanging the map on the wall.

Toots began the questioning.

Toots: Lady Shmendrick, when did you last see the diamonds?

Lady Shmendrick: Around midnight, just before I went to bed. I remember looking out of my window. It was just beginning to snow. All of the lights were out in the servants' quarters, so I locked the daimonds up in the safe and went to bed. The next thing I remember is waking up around 4 A.M. to the sound of alarms,and discovering my jewels were gone.

Toots: Thank you. It is clear from your testimony that since the lights were out before the snow started, the tracks were probably made by the thief ... Hanson, you're the chauffeur, what do you remember of last night's activities?

Chauffeur: I was sleeping soundly when I was awakened by the alarm. I was still in bed when I heard someone run into my house, then out the other door. I didn't get a chance to see who it was though. Sorry about that.

Toots: That's O.K. Hanson. Your testimony seems to fit the evidence on the map. (See map) There are two snow tracks connected to your cottage. The thief must have taken one of them going into your cottage and the other while running away.

Cook: I'd like to say something, Inspector.

Toots: Yes, Cromwell?

Cook: I couldn't sleep at all last night. I was thinking about a new dessert recipe that I've been working on. Chocolate Chip Mint Marshmallow Swirl is what I call it. Well, all of a sudden someone ran through my house and right out the other door. Then about five minutes later it happened again. I couldn't tell who it was, but I did see that the intruder was carrying the family jewels.

Butler: Well that's incredible! The same thing happened to me. In and out someone ran, then in and out again.

Maid: That happened to me too. I'm not sure how many times, but I know more than once.

Toots: One of you is a liar and I know who it is. (Only one person's testimony does not fit the evidence on the map; do you know who it is?)

First, let's consider the cook's testimony. THe cook said that the thief ran in and out of this house twice. This certainly makes sense. There were four tracks connected to the cook's house. The thief made one track going in and the other while leaving. Then the third and fourth tracks were made as the thief came and went the second time. Even though it is impossible to say which tracks were made when, still, four tracks account for two trips through the cook's quarters.

Maid: Well then, that gets me off the hook too, doesn't it? I also have four tracks leading to my house. So the thief must have been in and out of my house twice also.

Toots: That's correct.

(All of a sudden, everyone in the room turned towards the butler, who had only three tracks leading to his house. The butler began to shake and tremble.)

Toots: Well, Butler, the game is up. You only have three tracks to your house. If the thief ran in and out that would account for two tracks. But with only one more track left the thief must have gone into your house and stayed there. That means the thief is you!

Butler: That's right, Inspector, I did it. Last night before everyone went to bed I hid in the mansion instead of going home. Around 3:45 A.M. I stole the jewels from the safe and ran outside. When I saw all the snow on the ground I knew that I couldn't go straight home. So I decided to to make extra tracks in the snow. Naturally, I dragged my feet so you couldn't tell if the tracks were coming or going. When I thought I had made enough tracks, I went home. But, Inspector, you were too clever for me.

Toots: Officers, take him away. Lady Shmendrick, the case is solved. Ya know in the 23 years that I've been at Scotland Yard, there's been one thing I've always wanted to say.

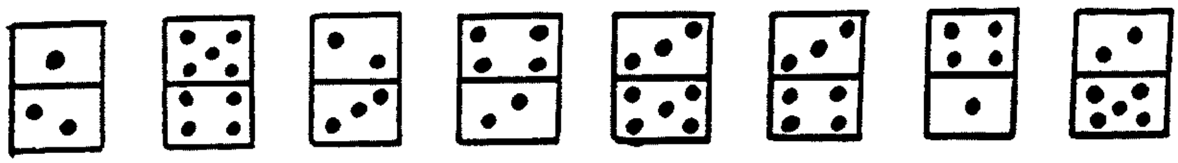
Lady Shmendrick: Yes, Inspector, what is that?

Toots: The butler did it.

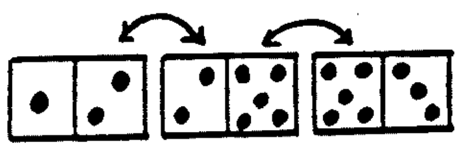
(Organ music up. Fade to black.)

EULER ACTIVITY #4: GOLDINI'S TRICK OF THE MONTH

<u>What You Will Need</u>: You must get eight dominoes with the exact numbers shown in the picture. You can either get them from a full set of dominoes or make them yourself with cardboard and crayons.



<u>THE TRICK</u>: Tell your friend that you will read his mind. Explain that later, after you turn your back, you'd like him to arrange all the dominoes in a row, but, just as in the <u>game</u> of dominoes, <u>numbers next to each other must match.</u> Show him what you mean. For instance.......



Now turn your back and have him arrange all eight dominoes in a row just as you have showed him. After he is all finished tell him to look at the two numbers at the ends of the row and concentrate on them. You will now read his mind. You pretend to concentrate very hard,, then finally announce that the two numbers he is thinking of are 3 and 5.

Now turn your back towards your friend, point to the 3 and 5 at the ends of the row, and take a bow.

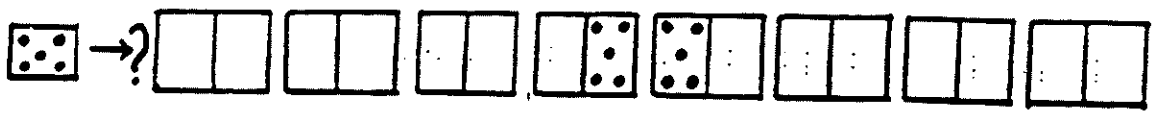
<u>THE NUMBERS WILL ALWAYS BE 3 AND 5</u>

You can't miss! Here's why.

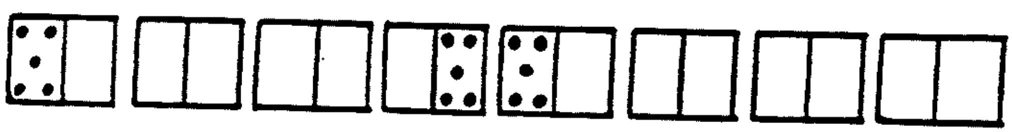
[GOLDSTEIN/Reynolds/Zangari]

Suppose a certain number shows up only in the middle of the chain. Then that number has to appear in pairs.

That means an even number of that number. What if there were a number that showed up only three times -- that is, an odd number or times?

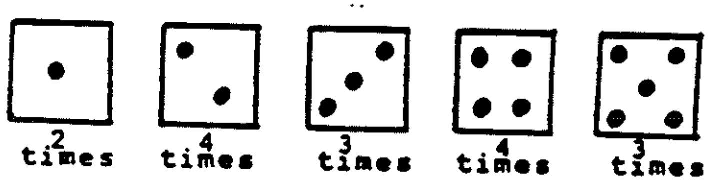


After putting in two of the 5's, where could you put the last one? You can't put it in the middle of the chain because that would break the rules. (Do you know why?) So the last 5 has to appear at an end of the chain.



Whenever a number shows up only three times, one of them must go on an endpoint!

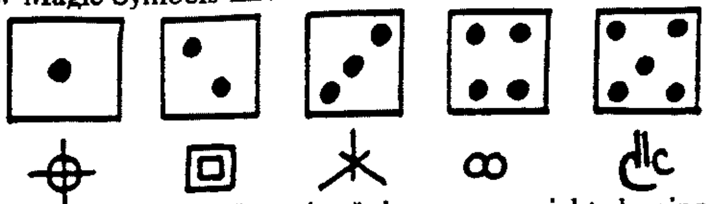
Now look at the eight dominoes we used in our trick. How many times does each number show up?



The 3 and 5 each show up three times so they must be at the endpoint of the chain. No matter how hard your friend tries, he has to put a 3 and a 5 the ends of the chain.

Pretty sneaky, huh?

If you like, you can make a set of dominoes with Magic Symbols. First make a chart with your new Magic Symbols filled in.



Now use your chart to "translate" those same eight dominoes that we used in the trick. You now have eight Magic Symbol cards.

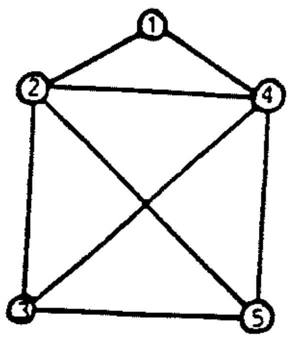
If you do the trick with this new set of domino cards, what two symbols will your friend be thinking of?

Naturally, you could make up a set of cards with colors, or pictures. Whatever you like.

Remember - Never Tell The Secret Of Any Magic Trick.

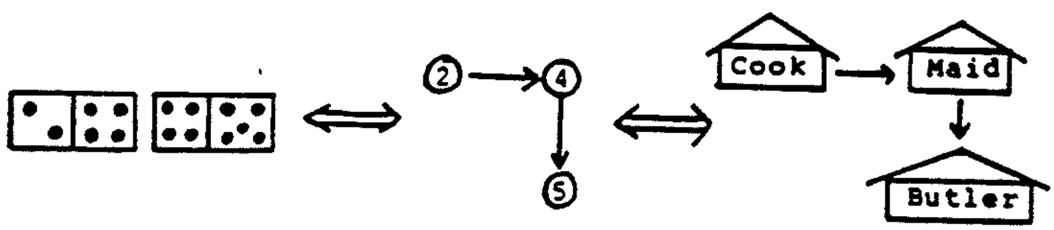
TEACHING NOTES:

Both The Case of the Stolen Diamonds and Goldini's Trick of the Month are examples of the same network tracing problems in topology first worked on by Leonhard Euler in the 18th century.



To construct the domino trick, each line of the network to the left is replaced by its corresponding domino. For example, the line connecting the 2 and the 4 is represented by the domino:

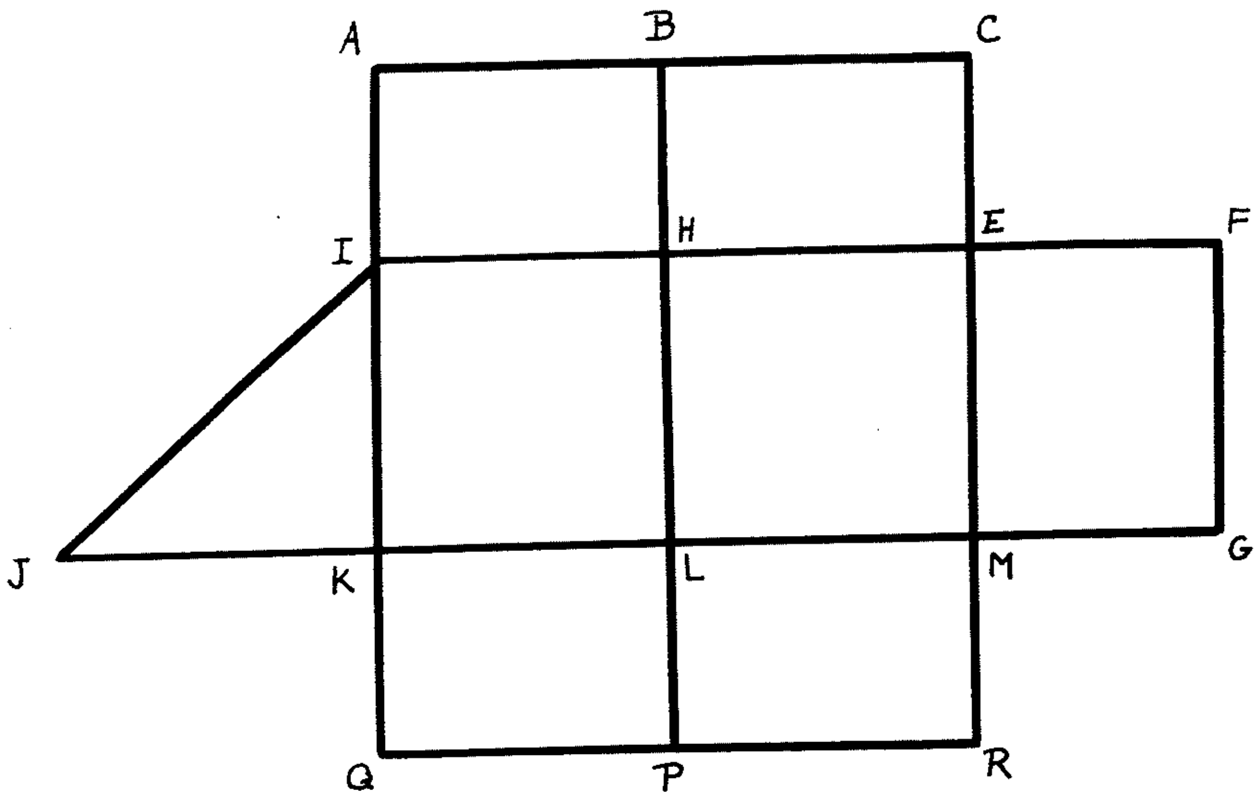
The requirement that adjacent numbers on the dominoes must match is tantamount to saying that one must not lift one's pencil while tracing the network, or that the thief cannot "fly" from house to house, but must leave tracks.



Each different chain of dominoes corresponds to a different way to "trace the network". Once this analogy is realized, then each statement in the Stolen Diamonds argument has its counterpart in the Dominoes explanation. For example, Inspector Toots' remark about four tracks accounting for two "in and out" trips by the thief is countered by the statement that a number that shows up in the middle of the domino chain must occur an even number of times.

EULER ACTIVITY #5: The Mail Carrier Problem

The figure is a map of a rural postal route where all the mailboxes are to be placed on one side of the street. Can you find an Eulerian path for the mail carrier so the mail is delivered without driving along any road more than once? Assume the office is located at vertex B.



[COZZENS/Johnson]

EULER ACTIVITY #6: A Discovery Lesson on Making a Non-Eulerian Graph Eulerian

Teaching Note:

Discrete mathematics offers the opportunity for students to actually DO mathematics. That is, not just solve problems, but explore and question beyond the usual presentation of concepts. Activity #6 is an example of such a lesson. The students develop new problems which may not be solved.

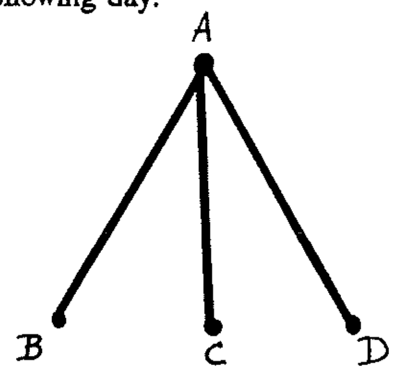
The Eulerian Theorem was presented and discussed. Next was presented the problem of the city planner whose job it was to find the most efficient way to plow the streets of a city, beginning and ending at the housing garage, in a city that did not have an Eulerian circuit. The problem thus became one of finding the minimum number of duplications of city streets and developing a formula to give this number for any graph not having an Eulerian circuit. Worksheet 1 was then assigned and students were directed to keep a log of questions that arose as they worked.

Reactions and Results

Many students realized that if they made the graph Eulerian, the problem would be solved. A discussion of the best way to do this followed. As they worked, most students hit upon the formula 1/2 (# odd vertices). This worked fine until they reached diagram #8. Frustration! Requests for the right answer or at least for the page in the book that would give them the right answer were denied. In desperation, they resorted to thought.

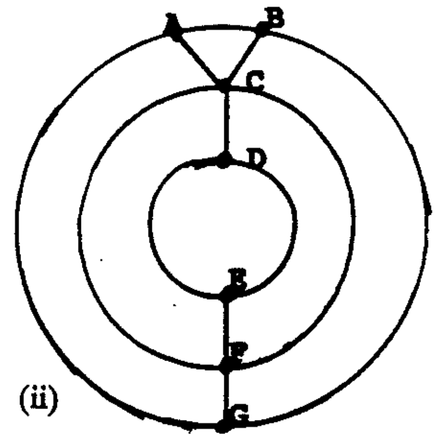
One class decided to modify their formula to read: 1/2 (# odd vertices) IF the odd vertices are connected to each other (as in diagram #1) OR (# odd vertices) if the odd vertices are not so connected (as in diagram #8). The assignment given for that night was to try to find an exception to the modified formula. Two exceptions, shown in Diagram 1, were presented by students the following day.

DIAGRAM 1



(i)

Beginning at A, this graph has 4 odd vertices and requires 3 duplications for the circuit ABACADA.



(ii)

Beginning at A, this graph has 6 odd vertices and requires 4 duplications for the circuit AGFEDEFCDCFGBCABA.

[Shyers]

Both diagrams provoked much discussion of the previously agreed upon definition of "connected to each other." Attempts were made to modify the definition, then to re-modify the formula to accommodate the two exceptions.

The other class decided to allow rectangular grids as the exception to the rule and set about finding a formula for the minimum number of duplications in rectangular grids of varying dimensions. Three possible cases were discussed and their formulas found:

(1) rows and columns both even
The formula decided upon was (# rows + # columns).

(2) rows and columns both odd
The formula initially used was (# rows + # columns) - 2.
Then it was noticed that 1/2 (# odd vertices) worked and so fit into the original formula.

(3) either rows or columns odd
(# rows + # columns) worked except when either value was 1 in which case 1/2 (# odd vertices) worked.

Their formula finally became: 1/2 (# odd vertices) except in a rectangular grid when # rows and # columns are both greater than 1 and either # rows or # columns, or both, are even, in which case the formula is (# rows + # columns)! Truly, a respect for mathematics is being gained when one must wrestle with the results of one's own creation.

The log of questions consisted, in part, of the following:

* How many different routes are there to be checked?
* Does it matter where we start?
* What if we just want a path and not a complete circuit?
* Are there other maps besides rectangular grids where the formula we are using does not work?
* How do I know if I've tried enough cases of the type of graph I'm using to get a formula?
* How do I know there aren't more graphs than the ones I've already thought of which might wreck my formula?

Any one of these questions could lead to discussion; some might be assigned as individual or group projects. Wherever the exploration leads, your goal, as teacher is to allow the questions to be asked. It may be better when you don't know the answers; then you are not tempted to lead the discussion and have the student "discover" your solution.

This is a new and difficult role for us. We are conditioned to lecture rather than to experiment. There is the lure of the tried and true, the ease of dealing with rules and procedures and students in straight rows. For us, as well as for our students, the making of mathematics is a new challenge.

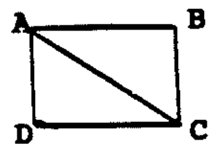
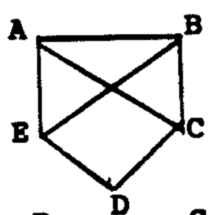
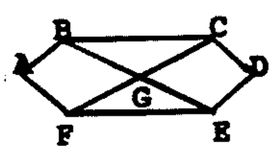
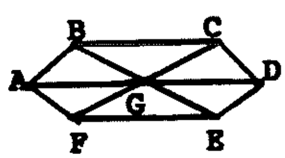
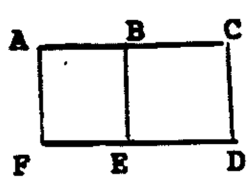
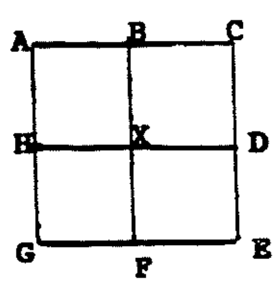
WORKSHEET 1

DIRECTIONS

***Find** the minimum number of duplications needed to trace each edge exactly once. Begin and end at point a.

***Show** the circuit by listing the edges.

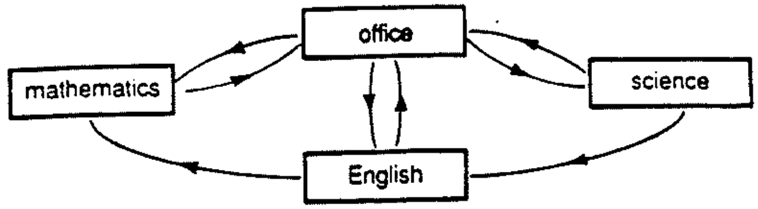
***Find** a formula that will allow you to predict the number of duplications needed.

		CIRCUIT	**# DUPLICATIONS**
(1)		------------------------	-----
(2)		------------------------	-----
(3)		------------------------	-----
(4)		------------------------	-----
(5)		------------------------	-----
(6)		------------------------	-----
(7)		------------------------	-----
(8)		------------------------	-----

SECTION II. DIRECTED GRAPHS

DIGRAPH ACTIVITY #1: Definitions and Models

At U-Rah High School the intercom lines are connected according to the graph at the right:



This graph is called a directed graph or digraph, with the arrows indicating the direction of the flow. Another method to show the same information is to use ordered pairs, where m = mathematics, e = English, o = office, and s = science.

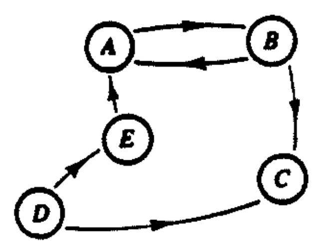
(m,o), (o,m), (o,e), (o,s), (s,o), (s,e), (e,o), (e.m)

A third method used to show this information or to record the ordered pairs is in an array called a matrix.

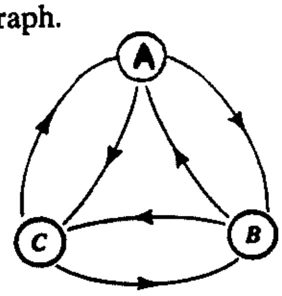
	m	s	o	e
m	0	0	1	0
s	0	0	1	1
o	1	1	0	1
e	1	0	1	0

Each entry in the array corresponds to the number of single arrows connecting consecutive vertices. A 1 appears in the (e,m) cell because an arrow connects English and mathematics.

1. What is the difference between (o,s) and (s,o)?____________________
2. Is it possible for the mathematics department to place a direct call to the English department? ____________________
3. How could the mathematics department call the English department?____________________
4. What does the 1 corresponding to the pair (e,o) in the matrix mean?____________________
5. In the diagram at the right, the symbol A B means that team A defeats team B. List the ordered pairs that describe the digraph.____________________



6. Write the matrix for the following digraph.



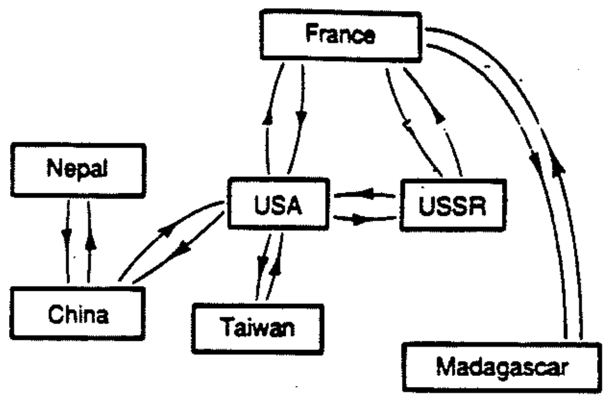
The editors wish to thank Jack and Gail Burrill, Whitnall High School, 5000 South 1116 Street, Greenfield, WI 53228, for writing this issue of September, 1988 NCTM Student Math Notes.

7. From information obtained in the digraph, use the ordered pairs in number 5 to create a matrix. For example, in number 5 we see the ordered pair (E,A). From the digraph we know that E defeated A. In the matrix in the row marked E and under column A, we assign a 1. If E had not defeated A, we would have assigned a 0. Complete the matrix at the right.

	A	B	C	D	E
A	0	1			
B			1		0
C					
D					
E	1				

8. Use the matrix in number 7 to answer the following questions:
   a) What is the win-loss record of team of team A?____________________
   b) If A is to play D, who do you predict to win?____________________
   c) What is the total number of wins in the conference?____________________
   d) Which team has the best record?____________________
   e) Which team has the worst record?____________________

At the right is a digraph concerning the diplomatic relations among certain governments.



9. How would the USSR communicate with Madagascar?____________________

10. How would Taiwan communicate with China?____________________

11. Which countries are most isolated?

____

Which country has the most diplomatic relations?

____

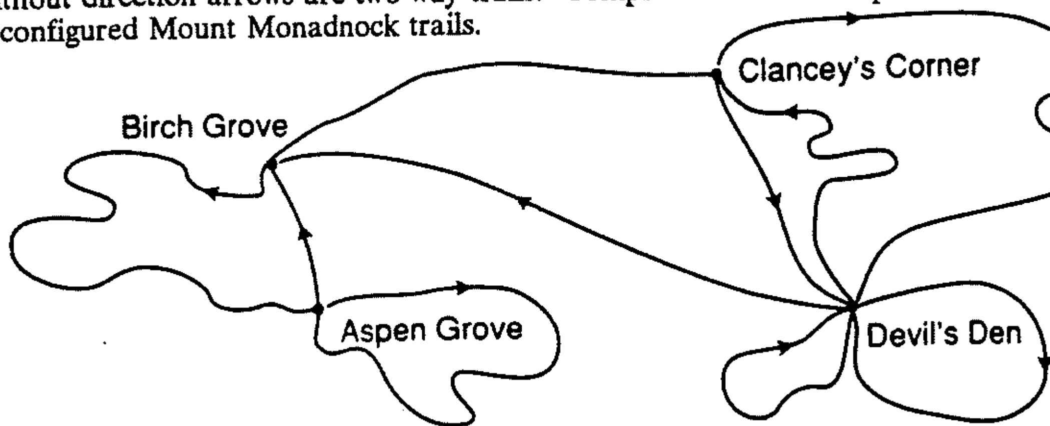
12. If every country communicated with every other country, how many ordered pairs would result?

____

____

DIGRAPH ACTIVITY #2: DEFINITIONS AND MODELS (Continued)

1. To reduce traffic and slow soil erosion on Mount Monadnock, park rangers have decided to designate some ecologically sensitive trails as one-way trails as shown below. For example, now just one way to travel from Aspen Grove to Aspen Grove is permissible--clockwise. Trails without direction arrows are two-way trails. Complete the matrix representation for the reconfigured Mount Monadnock trails.



2. The graph in the last exercise is called a directed graph because some lines (edges) representing the trails can be traveled in only one direction. Look at the matrix in the last exercise.

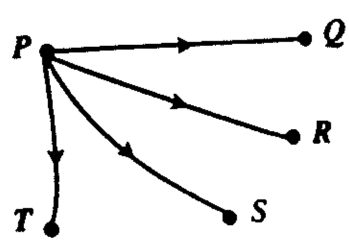
   a. Sum the elements in the third row. What does this number represent on the map? Sum the elements in the third column. What does this number represent on the map?

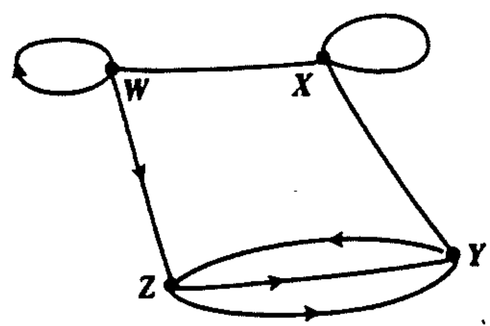
   b. Find the sum of all elements in each row. Find the total of these sums. How is this total related to the number of edges in the graph? Why?

   c. What type of integers appear along the major axis? Why?

   d. Does the major axis form a line of symmetry for the matrix?

3. Write the matrices corresponding to the graphs below.





From the *Mathematics Teacher*, February 1990

TEACHING NOTES:

A. Digraph has a Euler path if:

1. *The number of arcs entering each vertex is equal to the number of arcs leaving the vertex.*
2. *The number of arcs entering a vertex equals the number of arcs leaving the vertex for all but at most two vertices, and one of these two has one more coming into the vertex (end of Eulerian path) and the other one more outgoing edge (start of Eulerian path).*

ACTIVITY #2 SOLUTIONS:

1. $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

2. a. The sum of the elements in the third row is 3, which represents the number of ways one can leave Clancey's Corner. The sum of the elements in the third column is 2, the number of ways to get to Clancey's Corner. These sums are not necessarily equal in a directed graph.

2. b. The sums of the rows are 2, 2, 3, and 4, and the total of these is 11, which is the number of edges. Remember that the trail from Birch Grove to Clancey's Corner can be traversed in two directions, so it should be counted twice. Because this is a directed graph, the edges are not double-counted.

2. c. The elements along the major axis of a directed graph need not be even. The edges of a directed graph cannot necessarily be traversed in both directions.

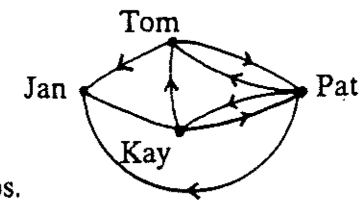
2. d. The matrix is not symmetric across the major axis.

3. a. $$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. b. $$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

DIGRAPH ACTIVITY #3: USING MATRIX MULTIPLICATION

Tom, Jan, and Pat all have modems for their computers. Their ability to speak to each other's computers is given by th directed graph shown. Find the number of different ways that Kay can communicate with Jan in two steps.



STEP 1. Write a Communications Matrix

TO

FROM C =		TOM	JAN	KAY	PAT
	TOM	0	1	0	1
	JAN	0	0	1	0
FROM C =	KAY	1	0	0	1
	PAT	1	1	1	0

STEP 2. Find the Product Matrix: C * C

$$C * C = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

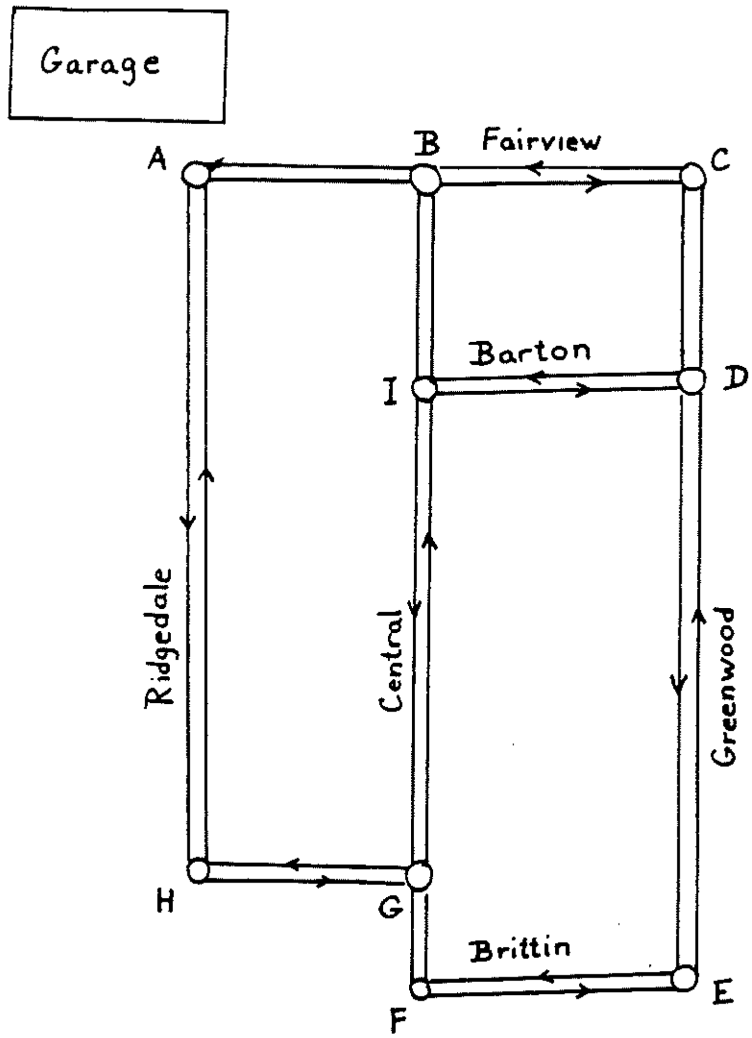
Kay can communicate with Jan two ways in two steps: K ---> T ---> J and K ---> P ---> J.

[Johnson]

DIGRAPH ACTIVITY #4: PLOWING STREETS AND GARBAGE COLLECTION

Present the problem of a snow plow needing to plow both sides of the street and every street in its area. Give the students a map and see if they can come up with a solution.

An answer: AHGFEDIBCDCBIDEFGIGHABA



Using the same map have the students either as a homework problem or working together as a class find a route for a garbage truck. The garbage truck needs to traverse each street only once. Find the distance it will take for the truck to collect the garbage and get back to where it started with the fewest duplications. An answer: AHGIGFEDIDCBIBA

[Simon, Walker, Coulter, Pasquale]

SECTION III: MATRIX REPRESENTATIONS OF GRAPHS

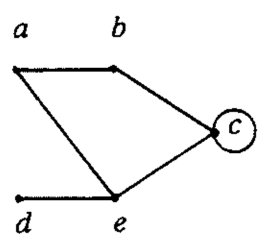
TEACHING NOTES: The Adjacency Matrix

Applications:
1. *Any "traveling salesman" problem.*
2. *The passing of toxins through the food chain.*
3. *Coding.*
4. *Communications.*

Adjacency Matrix. To obtain the adjacency matrix of the graph below, we first select an ordering of the vertices. Next, we label the rows and columns of a matrix with the ordered vertices. The entry in this matrix is 1 if the row and column vertices are adjacent and 0 otherwise. If $A = (a_{ij})$ *is an m x m matrix, then*

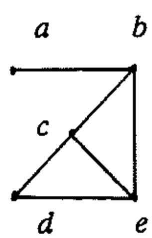
$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

Example 1.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} a\ b\ c\ d\ e \\ \begin{bmatrix} 0&1&0&0&1 \\ 1&0&1&0&1 \\ 0&1&1&0&1 \\ 0&0&0&0&1 \\ 1&1&1&1&0 \end{bmatrix} \end{array} = A1$$

Example 2.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} a\ b\ c\ d\ e \\ \begin{bmatrix} 0&1&0&1&0 \\ 1&0&1&0&1 \\ 0&1&0&1&1 \\ 1&0&1&0&0 \\ 0&1&1&0&0 \end{bmatrix} \end{array} = A2$$

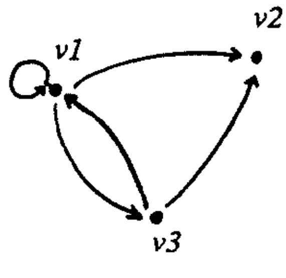
Note: We can obtain the degree of a vertex by summing its row or column.

While the adjacency matrix allows for the representation of loops, it does not allow for parallel edges. It is not a very efficient way to represent a graph, since the information, except that on the main diagonal, appears twice. Observe that $a_{ij} = a_{ji}$; *hence A is a symmetric matrix.*

[Csongor]

Matrix Representation of a Directed Graph (Digraph).

Both matrix representation and the concept of degree undergo changes in their applications to digraphs. In undirected graphs, the degree of a vertex is the number of edges "attached" to it. Since in digraphs edges have direction, we differentiate between the "in-degree" and the "out-degree" (the number of edges coming into or going out of the particular vertex from all other vertices).

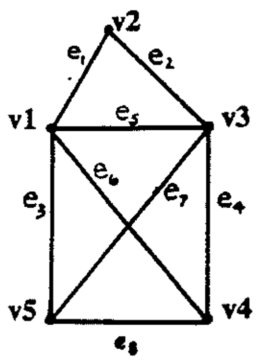


		v1	v2	v3
	v1	1	1	1
From	v2	0	0	0
	v3	1	1	0

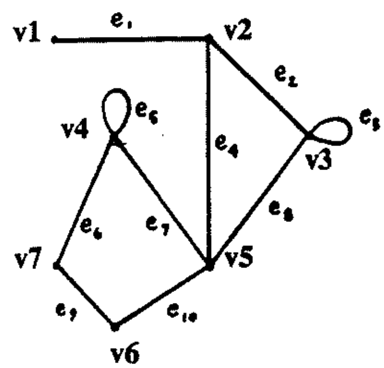
ADJACENCY MATRIX ACTIVITY #1

1. Write the adjacency matrix of each graph.

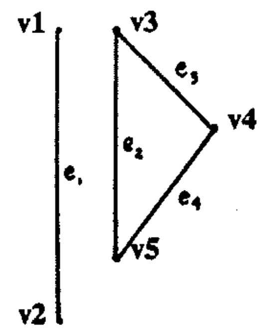
a.



b.



c.



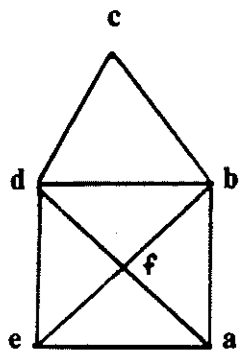
2. Draw the graph represented by each adjacency matrix.

	v1	v2	v3	v4	v5
v1	1	0	0	1	0
v2	0	0	1	0	1
v3	0	1	1	1	1
v4	1	0	1	0	0
v5	0	1	1	0	0

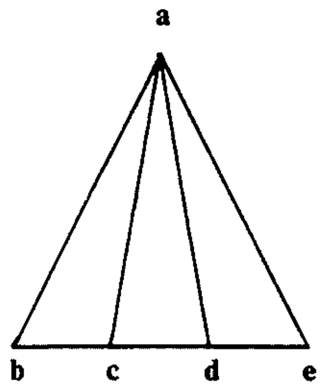
	v1	v2	v3	v4	v5
v1	0	1	0	0	0
v2	1	0	0	0	0
v3	0	0	0	1	1
v4	0	0	1	0	1
v5	0	0	1	1	1

ADJACENCY MATRIX ACTIVITY #2

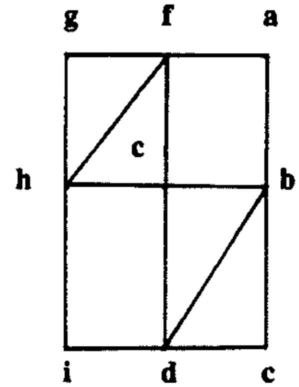
Using Adjacency Matrix to Determine If a Graph Has an Eulerian Path or Circuit:



I



II



III

1. Which of these graphs can be traced without removing the pencil from the paper and without retracing any lines?
2. Construct the adjacency matrix for each graph.
3. Add the columns in each matrix. What is each sum with respect to each vertex?

[COZZENS/Johnson]

SOLUTIONS:

I

	a	b	c	d	e	f
a	0	1	0	0	1	1
b	1	0	1	1	0	1
c	0	1	0	1	0	0
d	0	1	1	0	1	1
e	1	0	0	1	0	1
d	1	1	0	1	1	0
sum of each column:	3	4	2	4	3	4

The graph is traceable.

II

	a	b	c	d	e
a	0	1	1	1	1
b	1	0	1	0	0
c	1	1	0	1	0
d	1	0	1	0	1
e	1	0	0	1	0
sum of each column:	4	2	3	3	2

The graph is traceable.

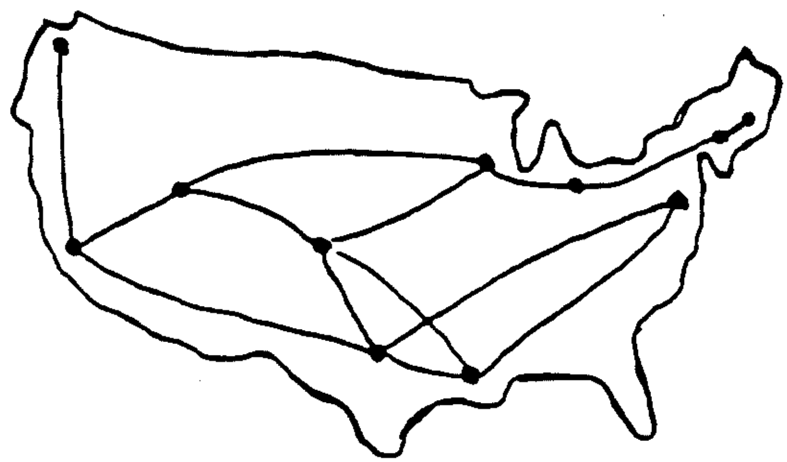
III

	a	b	c	d	e	f	g	h	i
a	0	1	0	0	0	1	0	0	0
b	1	0	1	1	1	0	0	0	0
c	0	1	0	1	0	0	0	0	0
d	0	1	1	0	1	0	0	0	1
e	0	1	0	1	0	1	0	1	0
f	1	0	0	0	1	0	1	1	0
g	0	0	0	0	0	1	0	1	0
h	0	0	0	0	1	1	1	0	1
i	0	0	0	1	0	0	0	1	0
sum of each column:	2	4	2	4	4	4	2	4	2

The graph is traceable.

ADJACENCY MATRIX ACTIVITY #3: MATRICES AND GRAPHS

Here is a "map" of the United States with eleven cities marked with dots. A certain airline has routes between some of the cities as shown by the route lins on the map.



The airline employed eleven new people to sit in the central towers of the eleven cities. The people are Alex, Barbara, Cindy, Donna, Elvis, Frances, Gloria, Hall, Irene, Johnny, and Karl. The two people in the cities with connecting routes will be talking to each other a great deal, to discuss the planes that fly from city to city. So, it would be helpful if these people were friendly to each other. Here are the pairs of people who are friends.

ALEX-BARBARA	HAL-FRANCES	IRENE-KARL	JOHNNY-IRENE
GLORIA-JOHNNY	GLORIA-IRENE	DONNA-ELVIS	JOHNNY-CINDY
DONNA-IRENE	ALEX-GLORIA	KARL-ELVIS	DONNA-KARL
CINDY-HAL	ALEX-DONNA		

Using matrices and explaining your algorithm, place the eleven people in the eleven cities so that the people in connecting cities are friends.

[Johnson]

TEACHING NOTES: GRAPHS, MATRICES AND CODES

Objective:

This module is for insertion in the second year of third year algebra curriculum. It is to show another use for matrices.

After completing this module, the students, given a problem, should be able to represent it in matrix and graph form. They will investigate possible methods of finding a solution and choose the best one for the situation.

Day 1

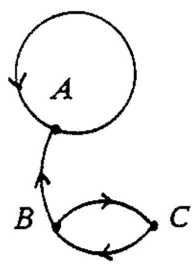
Information is presented and the students are to find a code word. The students are to be guided into trying to come up with a method to find the code word; setting up a matrix and a graph.

Example 1:

In the game of scrabble you are given 2 A's, 2 B's, and 1 C. You need to make a string of letters in such a way that:

B is followed by A
C is followed by B
A follows itself
B is followed by C

(Answer: BCBAA)



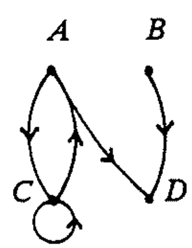
$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} A\ B\ C \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example 2:

Now find a code word such that the string of letters are to be placed:

A is followed by C
C follows itself
B is followed by D
A is followed by D
C is followed by A

(Answer: BDACCAD or ACCADBD)



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} A\ B\ C\ D \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

[Coulter,Pasquale,Simon,Walker]

GRAPHS, MATRICES, CODE WORDS ACTIVITY #1

Directions: For the data in problems 1 - 4, do each of the following:
a. Create a matrix; b. Draw a graph; c. Find a code "word".

1. The "word" uses 2 A's, 2 B's, 1 C and 1 D.
   B is followed by B;
   C is followed by B;
   B is followed by A;
   A is followed by D.

2. D is followed by A; B follows itself; A is followed by C; A is followed by B; B is followed by D; C is followed by A. (Hint: the number of A's, B's, C's, and D's is not the same as in problem 1.)

3. C follows itself; C is followed by B; A comes after D; D comes after C; B is followed by C.

4. C is followed by A; B is followed by C; C is followed by C; A is followed by D; A is followed by B; D is followed by C.

5. a. Given the matrix below, find the corresponding graph.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

   b. Suppose we change the matrix to the one shown below. What does the '2' mean? How does this change the graph?

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

GRAPHS, MATRICES, CODEWORDS ACTIVITY #2

1. Use the matrix below to find a codeword.

	A	B	C	D	E	F	G	H
A	0	2	1	1	0	0	0	1
B	0	0	1	0	1	0	0	0
C	1	0	1	0	0	2	0	0
D	1	0	0	0	0	2	0	0
E	0	0	0	1	0	0	0	0
F	1	0	0	1	0	0	1	0
G	1	0	0	1	0	0	1	0
H	1	0	0	1	0	0	0	0

(One answer is: CABCCFABEDACFDFGADGGDHAHDF)

2. Make up your own set of code "word" clues in matrix form to challenge your classmates.

TEACHING NOTES:

Use one of the problems from the previous day's homework to show how to use a matrix to come up with a code word. Talk about the 'ins' and 'outs' of the vertices by comparing the graph with the matrix.

Example 1, Day 1

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} A\;\;B\;\;C \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ 2\;\;1\;\;1 \end{array}\begin{array}{l} \\ 1 \\ 2 \text{ begin} \\ 1 \\ \\ \end{array}$$

end

Example 2

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} A\;\;B\;\;C\;\;D \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ 1\;\;0\;\;2\;\;2 \end{array}\begin{array}{l} \\ 2 \text{ begin or} \\ 1 \text{ begin} \\ 2 \\ 0 \\ \\ \end{array}$$

end

Solutions:

Example 1: *BCBAA*

Example 2: *BDACCAD or ACCADBD*

Give the class a large matrix (See Activity #2) to use to find a code word. Divide the class into two parts with each part solving the problem by either using the matrix alone (see Activity #2) or by drawing the graph first. The purpose is to show that a matrix can be easier to use than a graph for long code words. Also stress that a computer can only use a matrix.

SECTION IV: CONNECTED GRAPHS

TEACHING NOTES: *Introduction and Definitions*

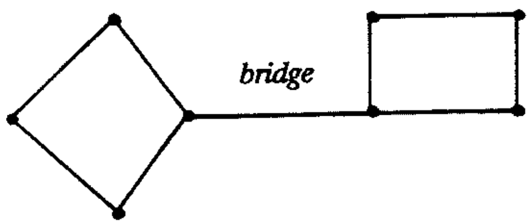
CONNECTED GRAPHS & THE ONE-WAY STREET PROBLEM

This unit is very appropriate for general math or less rigorous geometry class.

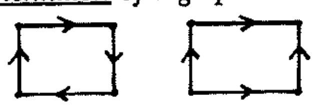
A graph is said to be connected if there exists at least one path from one vertex to any other vertex.

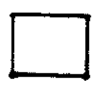
connected *disconnected*

In a connected graph a bridge is an edge which, if erased, will disconnect the graph.

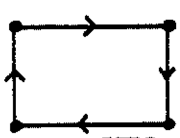


An orientation of a graph is an assignment of a direction to the edge of the graph.

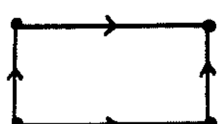


The above are both orientations of:  .

A digraph is strongly connected if there is a directed path from any vertex to any other vertex.



YES



NO

(If a graph has a bridge, it has no strongly connected orientation.)

THE ONE-WAY STREET PROBLEM

Think of a graph as the street map of a town. The local governing body wants to make all the streets one-way so as to reduce traffic jams or pollution. Of course, it is important that you be able to get from any one place to any other. Can this always be done?

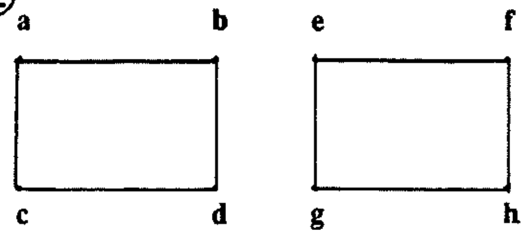
Translating this into mathematical terms, we are asking whether a graph has a strongly connected orientation. Certainly the graph has to be connected - if you can't get from A to B on existing roads, you won't be able to get from A to B on one-way roads. And certainly there can't be any bridges, because a one-way bridge will not permit you to go in the other direction.

It may be surprising, but any connected graph without bridges does have a strongly connected orientation! For such graphs, the one-way street problem can be solved.

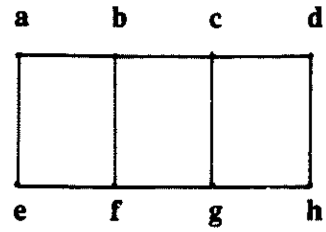
CONNECTED GRAPH ACTIVITY #1: CONNECTED GRAPHS AND BRIDGES

Determine which of these graphs are connected. Find all bridges.

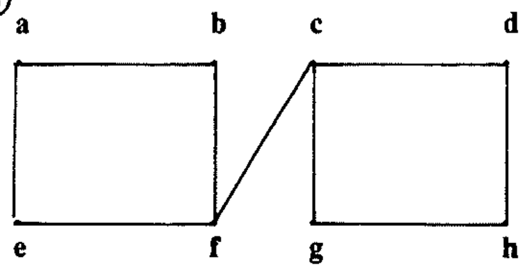
(1)



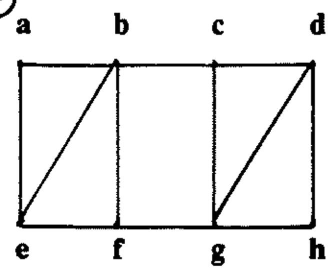
(2)



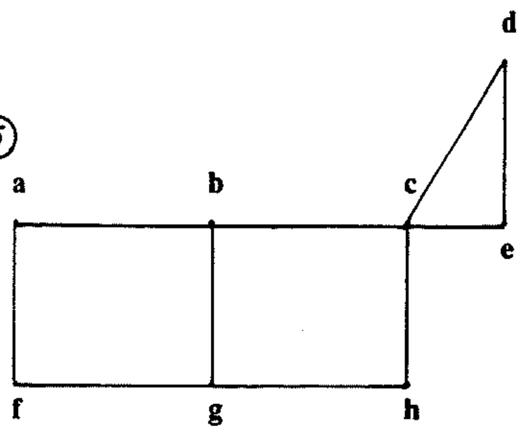
(3)



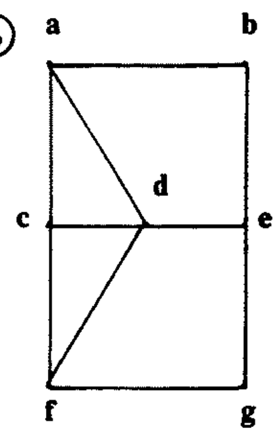
(4)



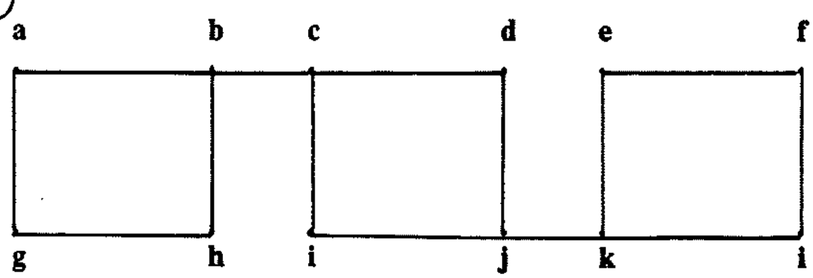
(5)



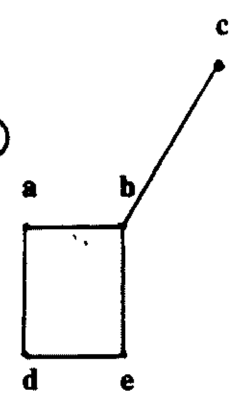
(6)



(7)



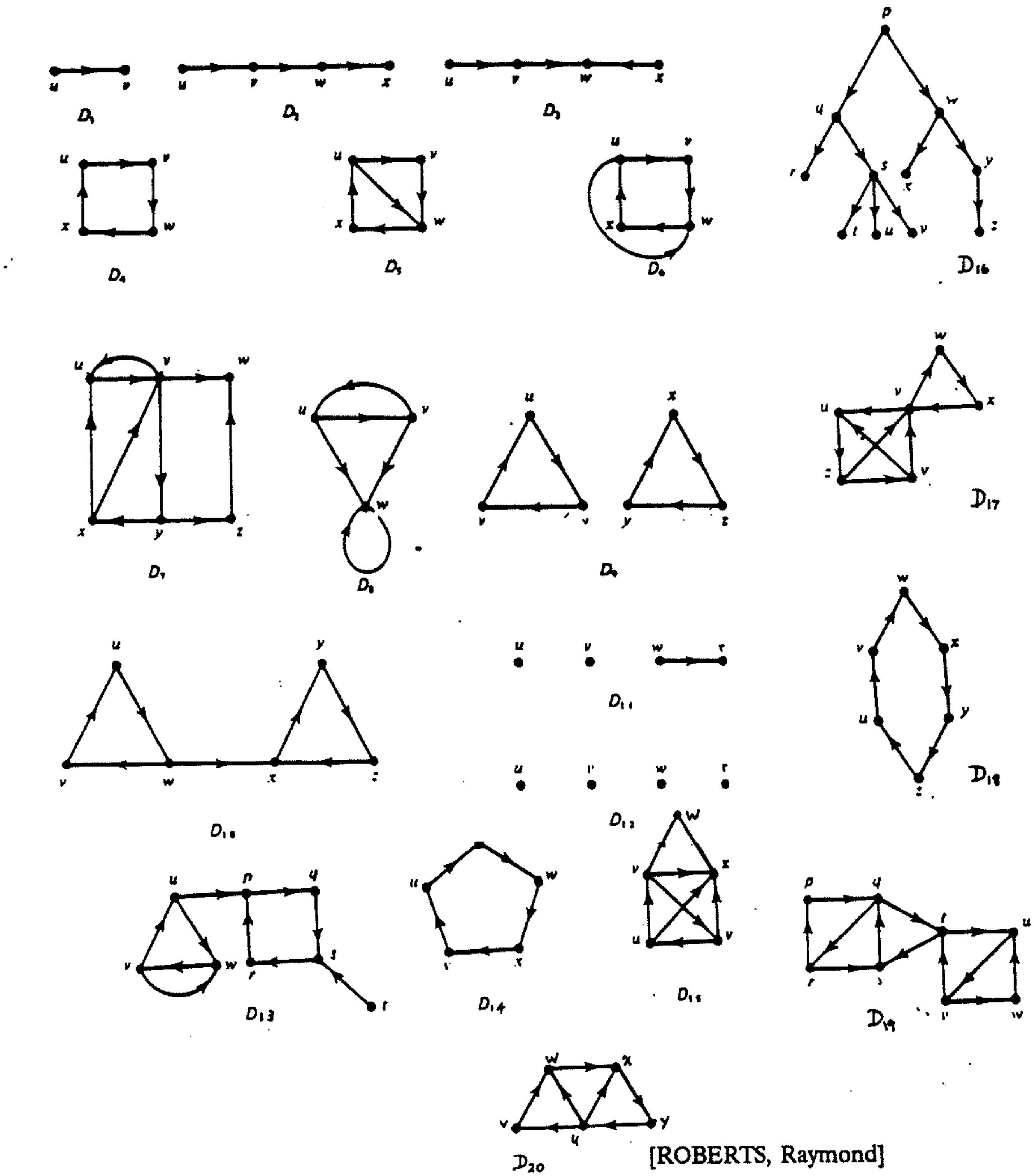
(8)



[Raymond]

ACTIVITY #2: STRONGLY CONNECTED DIGRAPHS

Determine which of the following digraphs are strongly connected.

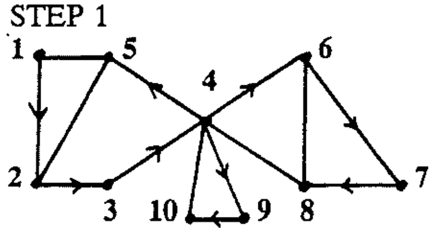


[ROBERTS, Raymond]

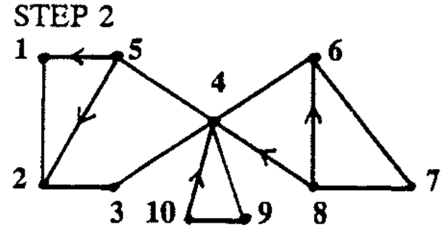
TEACHING NOTES: AN ALGORITHM FOR SOLVING THE ONE-WAY STREET PROBLEM

If a graph is connected and has no bridges, the following "depth-first" search can be used to impose directions on edges so that the resulting digraph is strongly connected. Walk through the graph numbering the vertices 1, 2, 3, ... and imposing the direction on the edges until you read a vertex all of whose neighbors are already numbered. Then back up and look for unvisited vertices. See the example in Step 1: Go from 1 to 2 to 3 to 4 to 5 imposing arrows from 1 to 2, from 2 to 3, from 3 to 4, and from 4 to 5. Then back up to 4, go from 4 to 6 to 7 to 8. Then back up to 4, go from 4 to 9 to 10. In Step 2, impose the opposite direction - from higher to lower number - on the remaining edges.

STEP 1



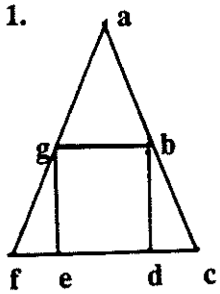
STEP 2



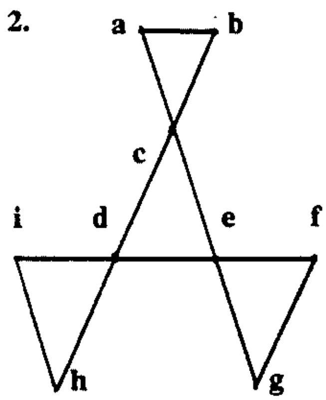
ACTIVITY #3: THE ONE-WAY STREET PROBLEM

Apply the one-way street method to these graphs:

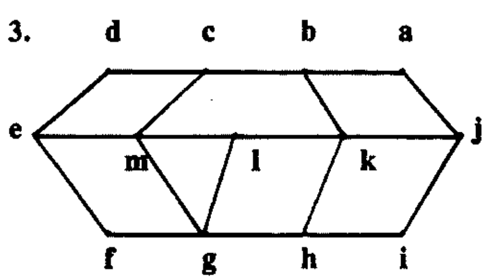
1.



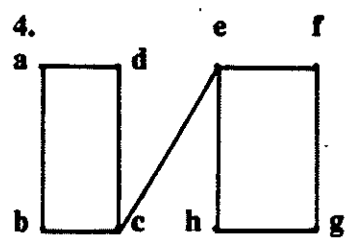
2.



3.



4.



[ROBERTS, Van Hise, Raymond]

SECTION V: MINIMUM SPANNING TREES

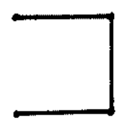
TEACHING NOTES: MINIMUM SPANNING TREES

In today's world, communications has become extremely important. Applications include telephone service, railroad computers, beepers, and airport towers.

A communications network can be modeled by a set of points connected with edges so that each point can communicate with each other and there are no cycles. This is called a spanning tree.

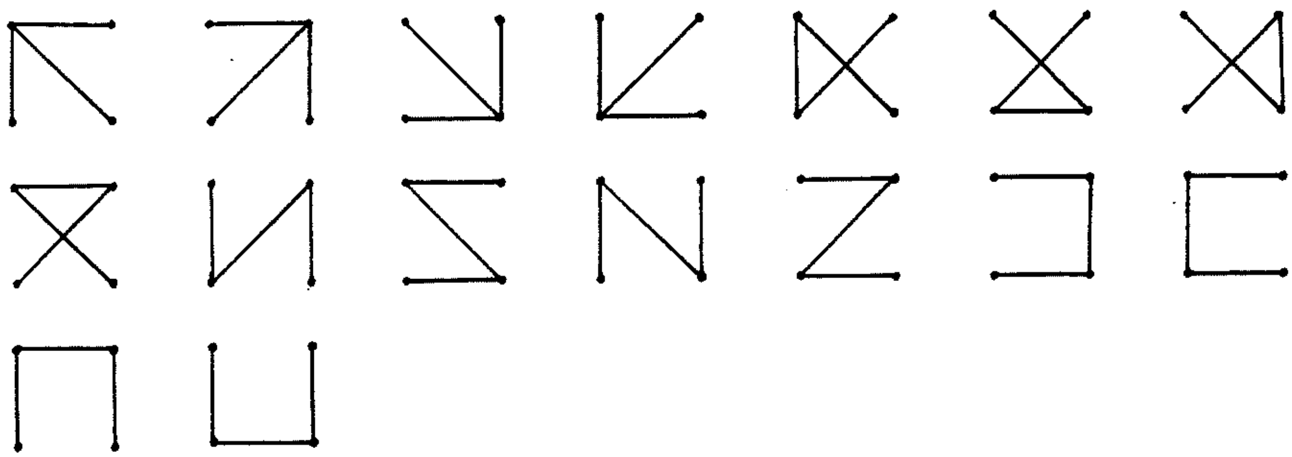
Example: In how many ways can 4 points be connected to form a spanning tree?

1. 

2. 

3. others?

Answer:



Question: How many edges are in all spanning trees on 3 vertices? 4 vertices? 5 vertices? n vertices?

Answer: All spanning trees on n vertices have n-1 edges.

Question: There is 1 spanning tree on 2 vertices, 3 different spanning trees on 3 vertices, 16 different spanning trees on 4 vertices, 125 on 5 vertices. How many different spanning trees are there on n vertices?

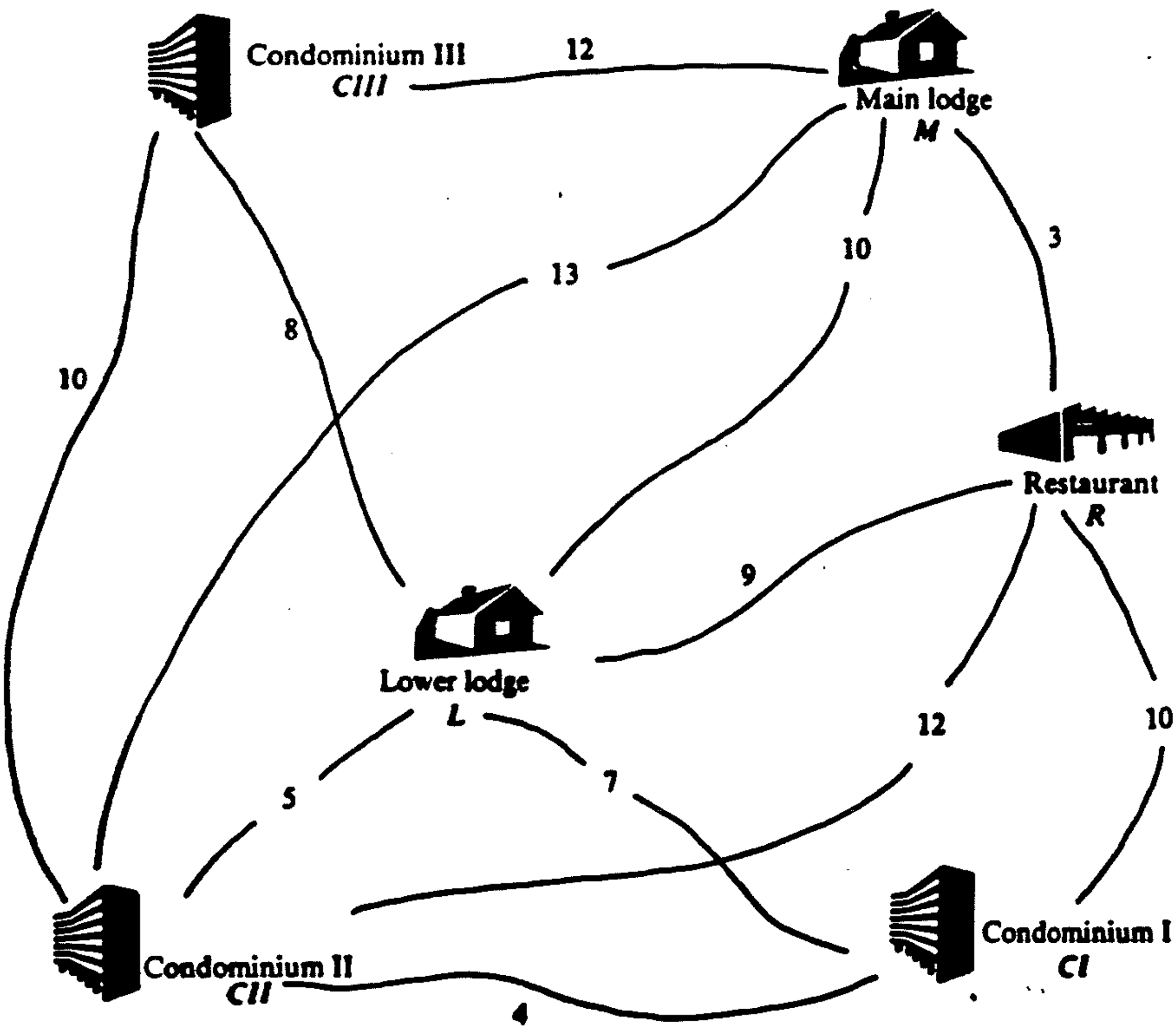
Answer: n^{n-2}

[Van Hise]

MINIMUM SPANNING TREE ACTIVITY #1: PLANNING ROADS FOR A SKI RESORT

A developer has purchased a 25-square-mile remote section of northern Vermont. There are no roads in the section, only a road leading into it. The developer will build a ski resort and condominium complex. The locations of the resort facilities and condominium units are fixed. The people can go from one location to another, not necessarily directly, within the area.

The construction firm that will build the roads has drawn the graph below of possible roads, accompanied by their costs in thousands of dollars. Find the smallest possible cost of building enough roads to connect the locations.



[COZZENS, Johnson]

TEACHING NOTES: MINIMUM SPANNING TREE

Give the students an opportunity to discuss the solution to the ski resort problem and develop a possible algorithm. What is meant by most economical? (Smallest possible cost to connect all locations in this example.) Since each edge is assigned a weight (cost in this case), this is called a weighted graph.

A connected graph without circuits is called a tree. A spanning tree is a subgraph of a graph that is a tree and includes all the vertices of the graph. When the graph is weighted, a tree of minimum total weight is called a minimum spanning tree.

Is CIII, M, R, L, CI, CII a spanning tree? (Yes)
Is it a minimum spanning tree? (No)
Is CIII, M, R, CI, L, CII a spanning tree? (Yes)
Minimum? (No)

The weight could represent time, distance, cost, etc. One strategy on deciding on roads is to make a list starting with the least expensive of the roads. Then choose the next least expensive of the roads until all locations are connected.

Road	*M-R*	*CI-CII*	*CII-L*	*L-CI*	*L-CIII*	*L-R*
Cost	*3*	*4*	*5*	*7*	*8*	*9*

Total cost = 3 + 4 + 5 + 7 + 8 + 9 = $36,000

This cost can be lowered by removing any one of the roads CI-CII, CII-L, CI-L. As an example, if CI-L is removed, the cost is 3 + 4 + 5 + 8 + 9 = $29,000. Here, the developer modified the strategy so that the addition of no edge would create a circuit; the result is the least costly.

An algorithm to find a minimum spanning tree based on the above above strategies is as follows:

Given a connected graph with weights on the edges:

(1) List the edges of the graph by increasing weights.
(2) Choose the first edge.
(3) Continue to choose the next edge as long as it does not create a circuit.
(4) Stop when the result is a spanning tree.

This algorithm was developed by J. B. Kruskal in 1956 for finding a minimum spanning tree. This algorithm is referred to as a "greedy" algorithm, since at each step an edge of minimum weight is chosen.

[COZZENS, Reynolds, Zangari, Johnson]

MINIMUM SPANNING TREE ACTIVITY #2: KRUSKAL'S ALGORITHM FOR LONG ISLAND

We started the class with a local map containing about 15 cities the students knew. We chose Nassau County and Western Suffolk County on Long Island and a copy of the map we used is enclosed. Try to make the map to scale.

Define a network and ask the students to attempt to connect the cities on the map, to come up with a minimum network. The results will be astounding. Then give out a mileage chart and ask the students to calculate the total mileage of the network. This is most easily accomplished if the students place the mileage along each edge of their graphs. A class comparison of the totals will be a real eye opener.

At this point, Kruskal's Algorithm can be described and the students can be given another map to fill in for homework using the mileage chart and the algorithm. This should end day 1.

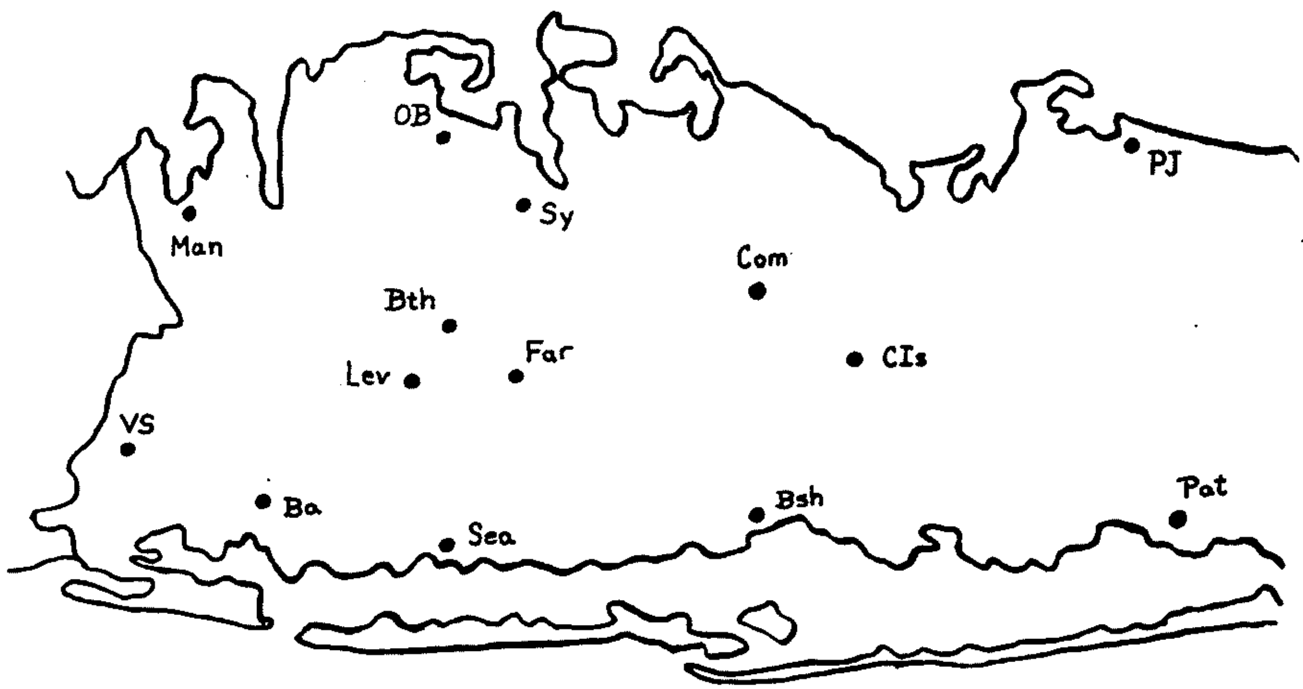
For day 2, the homework can be checked and students can see if Kruskal's Algorithm gives a better result than their first attempt. It can even be checked against the best answer from day 1. At this point, an optional topic would be to describe Prim's Algorithm* and apply it to the same map, comparing results. It should also be apparent to the students that some "central" points act as "junctions" and tend to minimize the network length. This discussion will prepare the students for a following discussion of Steiner Points. The teacher can spend as much time as needed to make the students comfortable with both algorithms. This should end day 2 (or day 3 if practice is needed).

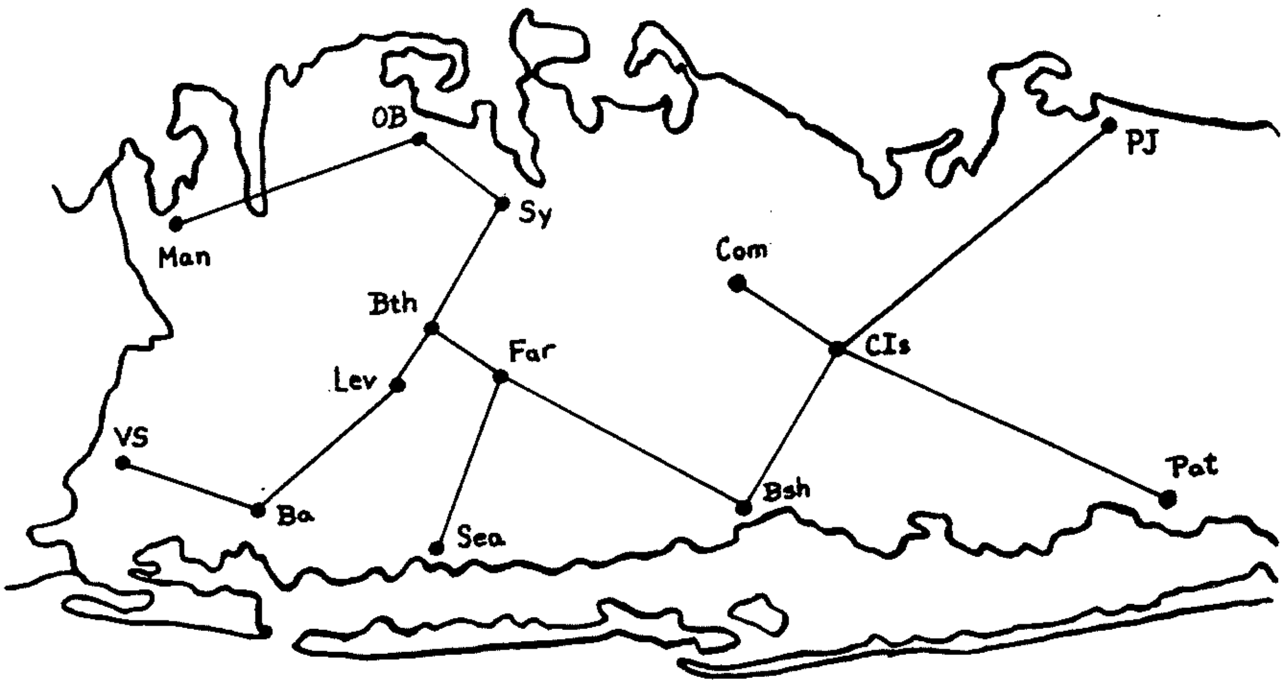
* Prims Algorithm will be described in a later edition.

[Schumacher, Eldi]

	Ba	Bsh	Bth	CIs	Com	Far	Lev	Man	OB	Pat	PJ	Sea	Sy	VS
Ba	-	20	9	24	21	11	6	12	16	32	32	7	11	5
Bsh	20	-	12	7	9	10	15	24	18	13	19	13	15	24
Bth	9	12	-	15	12	2	3	12	10	24	26	5	8	13
CIs	24	7	15	-	6	13	15	26	18	10	13	17	16	28
Com	21	9	12	6	-	11	15	22	13	16	14	16	11	25
Far	11	10	2	13	11	-	4	14	11	22	25	5	8	14
Lev	6	15	3	15	15	4	-	11	10	27	28	5	8	10
Man	12	24	12	26	22	14	11	-	10	36	35	15	11	10
OB	16	18	10	18	13	11	10	10	-	28	25	15	4	17
Pat	32	13	24	10	16	22	27	36	28	-	13	25	26	37
PJ	32	19	26	13	14	25	28	35	25	13	-	31	22	38
Sea	7	13	5	17	16	5	5	15	15	25	31	-	11	11
Sy	11	15	8	16	11	8	8	11	4	26	22	11	-	15
VS	5	24	13	28	25	14	10	10	17	37	38	11	15	-

KRUSKAL'S ALGORITHM FOR LONG ISLAND'S NASSAU COUNTY AND WESTERN SUFFOLK COUNTY

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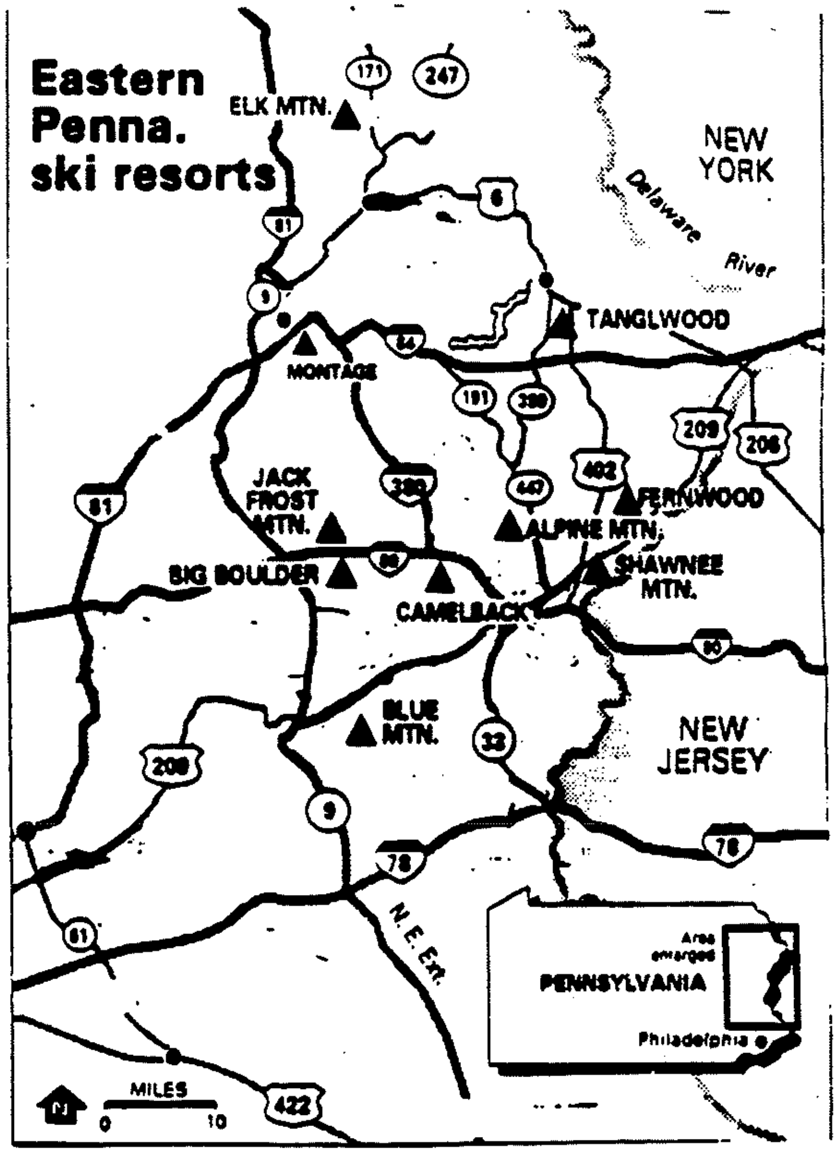
SECTION VI: SHORTEST ROUTE PROBLEMS

SHORTEST ROUTE ACTIVITY #1

In today's world, information processing has become as important as product. Finding "a" solution to a problem isn't good enough. Most real world problems do have more than one solution! How to find the best solution is the problem, the problem of today's mathematicians. They refer to it this way: "What is the most efficient algorithm?"

Example

Steve and Bunny Skibum want to visit every ski resort in the Poconos. Of course, they want to spend as much time on the slopes as possible, so they want to minimize the time they spend driving. How can the Skibums find the shortest route?



[Van Hise]

MILEAGE CHART

	Alpine	B.Boul	Bl. Mt.	Camel	Elk	Fernw.	Jack F.	Mont	Shawn.	Tanglwd
Alpine	0	25	35	20	50	17	32	30	12	22
Big Boulder	25	0	23	9	52	30	3	32	24	44
Blue Mtn	35	23	0	25	77	37	25	47	32	55
Camelback	20	9	25	0	52	26	11	27	20	36
Elk	50	52	77	52	0	43	50	25	49	28
Fernwood	17	30	37	26	43	0	34	40	6	26
Jack Frost	32	3	25	11	50	34	0	28	26	46
Montage	30	32	47	27	25	40	28	0	42	23
Shawnee	12	24	32	20	49	6	26	42	0	32
Tanglewood	22	44	55	36	28	26	46	23	32	0

TEACHING NOTES: SHORTEST ROUTE

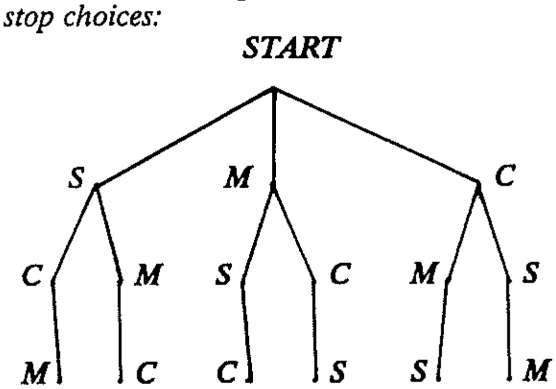
How many different tours of the Pocono ski resorts can the Skibums use? They can start anywhere and end anywhere but must visit each mountain exactly once. There are many possibilities. Here are two of them.

Tour 1	*or*	*Tour 2*
Shawnee		*Elk*
Blue Mtn.		*Tanglewood*
Big Boulder		*Fernwood*
Jack Frost		*Shawnee*
Camelback		*Camelback*
Alpine		*Big Boulder*
Fernwood		*Jack Frost*
Tanglewood		*Alpine*
Elk		*Montage*
Montage		*Blue Mtn.*
185 miles		*201 miles*

How many tours are there altogether in these ten sites? Before trying to answer this question directly, we will...

<u>*Consider a smaller problem first.*</u>

Suppose there are only 3 mountains, Shawnee, Montage, and Camelback. There are 3 choices for the first stop. Suppose we choose Shawnee, then there are 2 choices for the second stop: Shawnee, Camelback, or Shawnee, Montage. In each case there is only one choice for the third stop: Shawnee, Camelback, Montage or Shawnee, Montage, Camelback. However, we need to repeat this reasoning for each of the other first stop choices:



Therefore, for 3 mountains there are 3x2x1 = 6 tours from which to choose. If the Skibums are going to check the length of each possible tour for all ten mountains, they will have to check this many tours.

10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3,628,800 !!!

If it takes them one minute to check each tour, will the ski season be over before they are finished?

If there were 26 mountains, to find the shortest tour by checking all possible - 1.55×10^{25} - can you guess how long it would take a computer that can check 1 billion routes per second? <u>One-half billion years.</u> This illustrates the need to find algorithms providing solutions to this type of problem in a reasonable amount of time.

TEACHING NOTES: SHORTEST PATH

INTRODUCTION-

I experimented with two different ideas and two entirely different level classes. One class was a Math II. These are students who have passed the HSPT and are not going to college. The other class is a Probability and Statistics class made up of three students who have completed the Calculus course and want to continue their math studies. I will separate the classes in my descriptions.

Math II - The first day was a half day and instead of giving out the texts and spending all of the time on clerical information, I obtained an overhead projector and started in with each student giving me their name and where they spent their vacation. I drew a map of the United States on a transparency. (I decided not to use the maps in the phone book to get a perfect map, because I intended to ask each student to draw a map on an unlined piece of paper and I didn't want any student to feel that they couldn't do the assignment because they couldn't get a good enough map. My map was very rough at best.) Each student took turns giving me the place that they had visited. I placed a dot at the place that they named and then put their initials next to the dot. There were two students that hadn't gone anywhere, so they had their initials at the home town. Several had only gone to the beach for a day; I said that the beach certainly counted, so we placed the different shore resorts on the map. It turned out that everyone had been on the East Coast, and the West Coast was without dots. I found out a lot about each student as we listened to their vacation spots. It was interesting that one student had gone camping in New Hampshire, but had no idea where in New Hampshire. So he picked a place in the state. We had 10 students and I was flexible; if a student went to two places I put both up. They then had to copy the map as best they could making sure that they had each vacation spot marked. Their homework was to plan a trip so that they would start at our home town and visit each of the vacation spots just once, and return home. They needed an ordered list of the vacation spots.

DAY 2 and 3 - One person dropped the class and another one was added. We added the new vacation spot to our list and proceeded to compare the ordering of the vacation spots. This time I was prepared with maps of all sorts, some of the entire East Coast, some of the United States, some of just one state. Yes, I emptied out my glove compartment. Everyone knows the State test has maps on it. I was amazed at the lack of knowledge in using a map. My goal was to have each student find all of the distances from all of the cities and see which way was the shortest. I changed it to just find out how long the path that you choose is and we compared the distances. Beware, I had the distance between the same cities vary by hundreds of miles. We had to have a discussion on estimation, using a scale, and we decided to measure the distance as the bird flies. I didn't let them use a ruler, but rather the edge of a piece of paper like they may need to do on a state test. We compared our result and decided on a shortest path of those found and I assigned for homework that they list all of the possible paths. (I hoped that someone would ask how many there were, but no one did.) I began to explore how many different possibilities there would be with 12 places to visit. They then decided that it would be much too large an assignment, so after showing them how many different paths there would be with 2 places, then with 3 places, their homework was to list all of the ways you could visit 5 places. One student stayed around after class and said that he knew a teacher that could figure it out without listing them. He actually remembered and put 5 x 4 x 3 x 2 x 1 on the board before he left. I told him that if he knew how many there were, he would not have to list them all.

DAY 4 - Each student shared how they wrote down the towns. Some were very haphazard and others used patterns to list all of the paths. I had each student write a paragraph on how they would find the shortest path given any map. I let the student who had figured it out mathematically explain it to the

[Edwards]

class. Then we made a two column chart containing the number of cities to visit and the number of different paths possible. The discussion also included some of the paths with the same length. (Either a coincidence or the same path backwards.)

PROBABILITY AND STATISTICS - This class did a map of the colleges that they had visited. They then had to find the shortest path to visit all of the colleges on one big trip across the United States. They had places scattered in the east, west and Great Lakes region. They seemed to form an obvious shortest path. The written method for choosing the shortest path was very easy. Next time I would add two places that would change drastically their choices. (I may revisit this problem again this year if I can get the soap bubbles props. This is a good intro into the soap bubbles.) These students knew the formula for the number of different paths and had no difficulty with the maps. They had to get the mileages for homework. They did have trouble with writing down the method for finding the shortest path. It was the first time that they had written in math class, and I pointed out that some of the explanations did not always find the shortest path, but a path. This took two days.

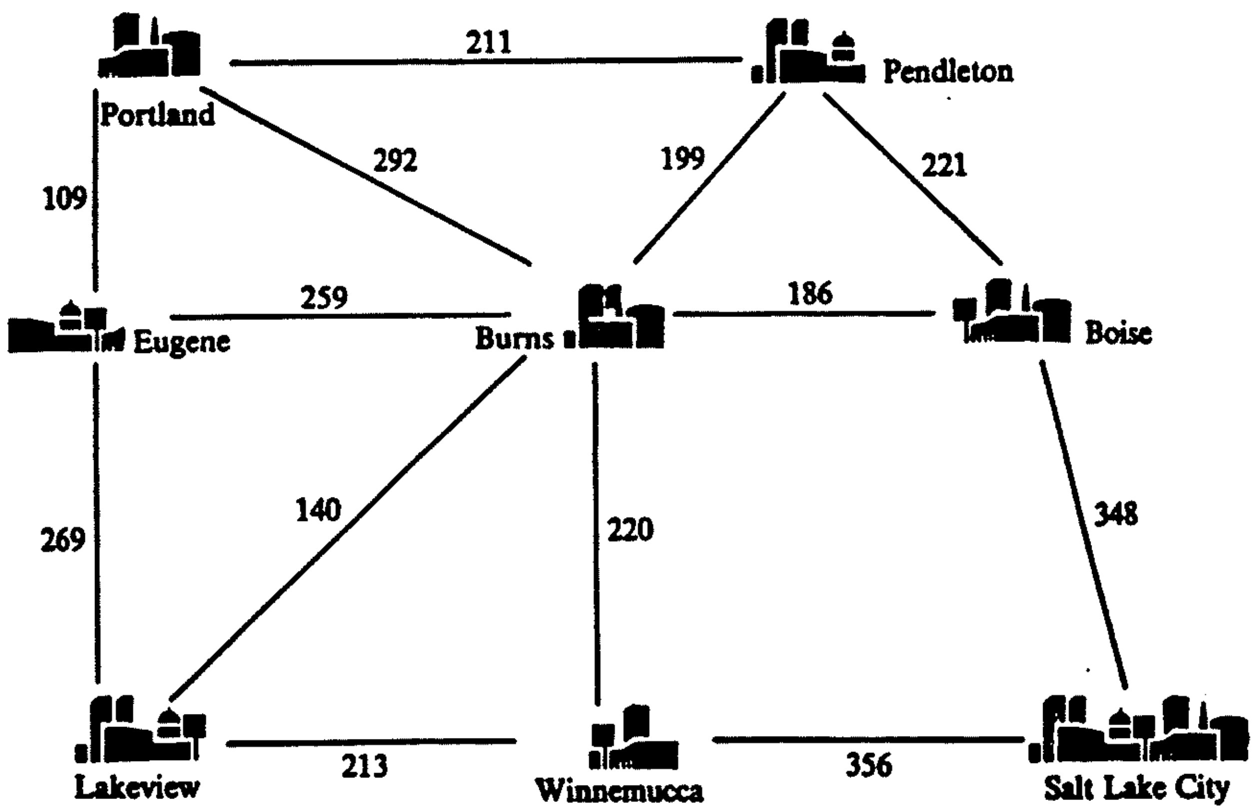
ADVANTAGES - Starting the year active the first day, doing something that a newcomer could join right in. I felt that I knew a lot about how they were thinking and how much they knew. Everyone felt like they had a part in the class. One student asked when he was going to get the text book, this was too much thinking. He just wanted to do some problems. One of the prob-stat students said "Oh, this is work." He was enjoying finding out where everyone had been. There was such an easy transition he hadn't realized that he was now utilizing math. This way I did not have to issue the text books until after the classes were settled. Everyone was there for the course requirements and issuing of texts.

PROBLEM 1 - The Probability class was complaining about the parking lot. The one way out in front of the school had been changed to one way in. They did not like the change. So we made a map of the roadways, simplifying them a bit, and their assignment was to consider any of the non-dead-end (feasible) street assignments. The students decided on the simplified map. It was interesting to hear their discussion of what was almost equivalent when they simplified the map. The task was to come up with a better plan with reasons or to substantiate the change with reasons. The students picked various parking places and relayed the advantages or disadvantages of the different possibilities. They decided that when school lets out there is no good way to get all of the cars out of the lot at the same time, but that the new street design is a good one. I haven't heard any complaints about it since. They decided that they were reacting to chance, not practicality.

PROBLEM 2 - The same week there was a complaint that it was impossible to get upstairs after homeroom. All of the students with homerooms upstairs are using the middle staircase to come downstairs and the students that are downstairs can't move through the T hallway at the bottom of the stairs. It is like all of the students are in one hallway at the same time. The traffic comes to a dead stop. I told them that if they could find a better way of circulating the students in the hallway I would take it to the administration. (I had already heard that they were aware of the problem and were looking into it.) We drew the school hallways and the students tested the advantages of one way stairs and/or hallways. They discovered a lot of problems. They settled on picking one class and comparing the different distances that one had to move to get there. They did this for several different maps. The final proposal was that there is no best way because none of them wanted to be forced to go out of their way to get to a nearby room. So no changes were proposed.

SHORTEST ROUTE ACTIVITY #2

Starting at Portland, Oregon, find the shortest route to Salt Lake City, Utah, along the given routes.



[COZZENS]

TEACHING NOTES: SHORTEST ROUTE

Finding the shortest route from one town to another and finding the least expensive airline route from one city to another are examples of shortest-route problems. Shortest route problems arise in situations where paths in a graph correspond to strategies for completing some task.

AN ALGORITHM DEVELOPED BY E.W. DIJKSTRA TO FIND THE SHORTEST ROUTE ON A DIRECTED GRAPH.

Check the following parameters first:

- *[1] The graph must be a di-graph.*
- *[2] The graph must have a starting vertex.*
- *[3] Each edge must be assigned a nonnegative weight.*

THE ALGORITHM

STEP 1 — *Circle the starting vertex. Underline the weights on each edge directed from the starting vertex to any uncircled vertex.*

STEP 2 — *From the edges directed from a circled vertex to an uncircled vertex, choose an edge with the smallest underlined number.*

- *a) Mark the selected edge with hash marks.*
- *b) Circle the uncircled vertex of the selected edge.*
- *c) Put a copy of the underlined number next to the newly circled vertex and circle the number.*

STEP 3 — *Put an underlined number on each edge directed from the last vertex circled to an uncircled vertex. The underlined number is the sum of the original number on the edge and the circled number near the circled vertex of an edge.*

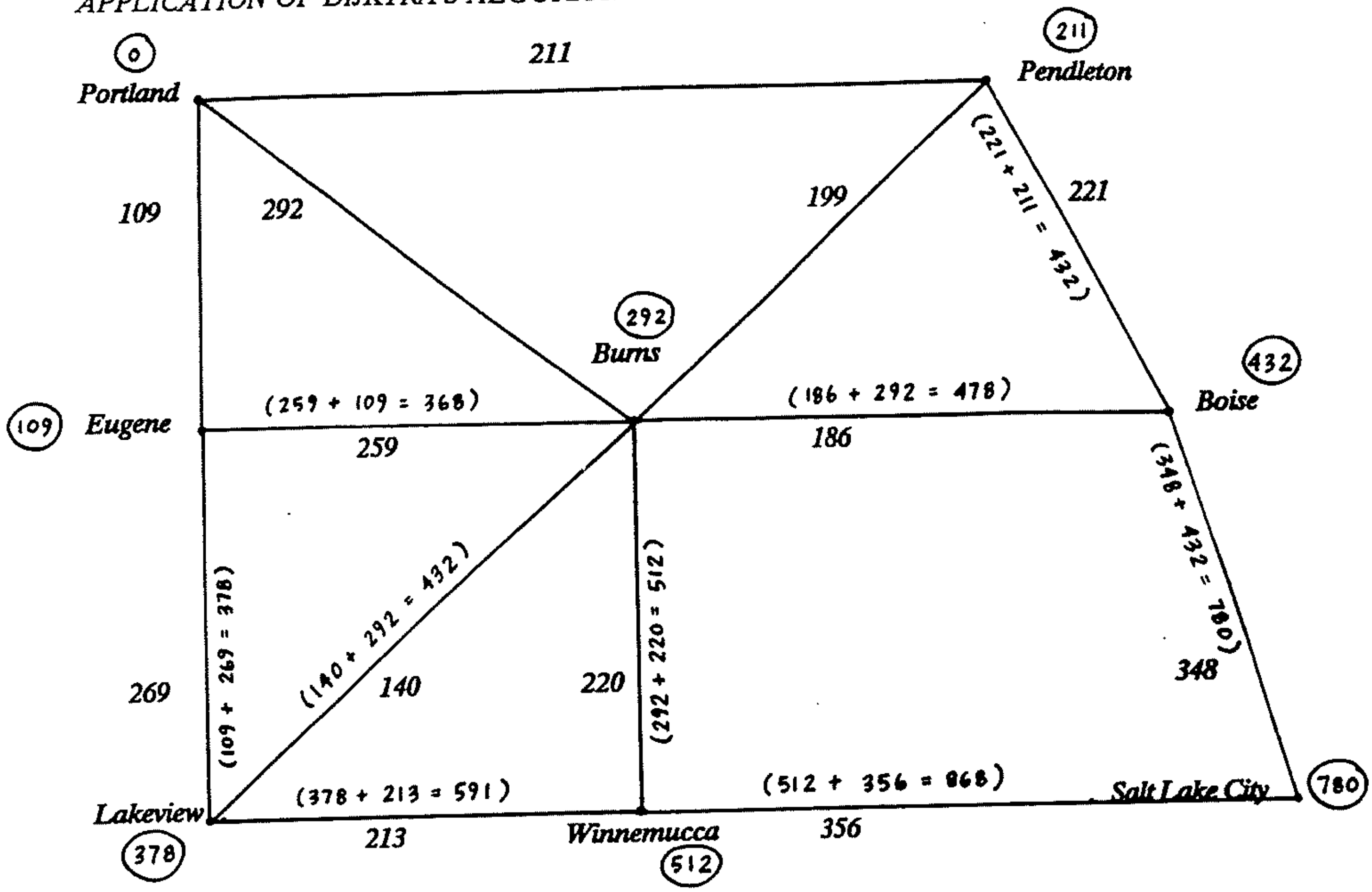
STEP 4 — *Repeat steps 2 and 3 until there are no edges directed from a circled vertex to an uncircled vertex.*

Now look for a connected graph from the starting vertex to the ending vertex. Add the weights along these edges and translate into a solution to the problem.

[COZZENS, Johnson]

TEACHING NOTES:

APPLICATION OF DIJKTRA'S ALGORITHM TO SHORTEST ROUTE ACTIVITY #2



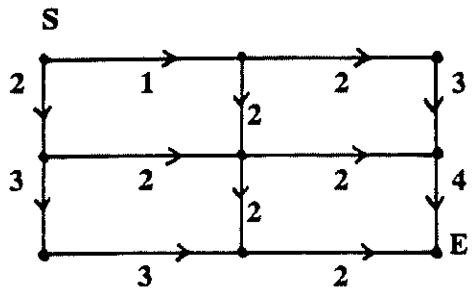
The edges are chosen and the vertices circled in the order of their distance from the starting vertex: Eugene, Pendleton, Burns, Lakeview, Boise, Winnemucca, Salt Lake City. The shortest route from Portland to Salt Lake City would be Portland, Pendleton, Boise, Salt Lake City (780 miles).

[COZZENS, Van Hise]

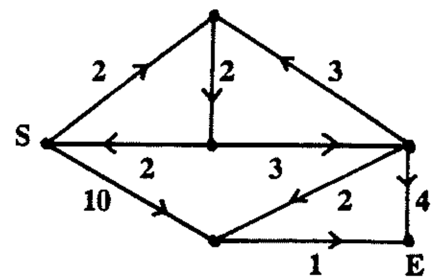
SHORTEST ROUTE ACTIVITY #3

In each graph find the shortest route from S to E.

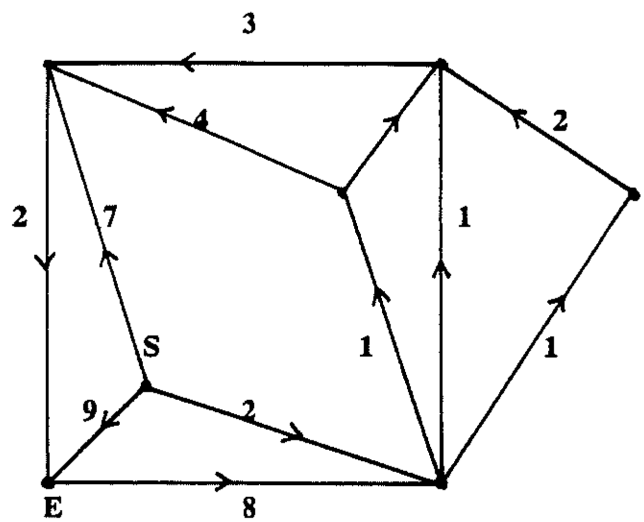
[1]



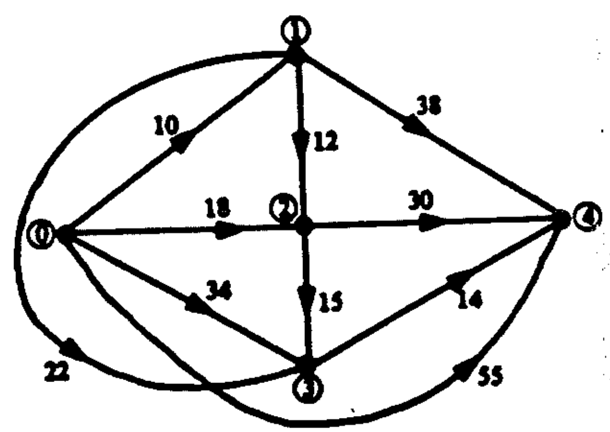
[2]



[3]



[4] After buying a tractor this year, a farmer will have the option next year and each year after of trading in his tractor for a new one or keeping the tractor for another year. The weights on the edges of the graph below give the cost to the farmer in thousands of dollars of buying a new tractor at the year indicated on one vertex of the edge and trading it in at the year on the other vertex. Find the best buy-sell strategy for the farmer if he plans to sell his farm and not have a tractor at the end of:

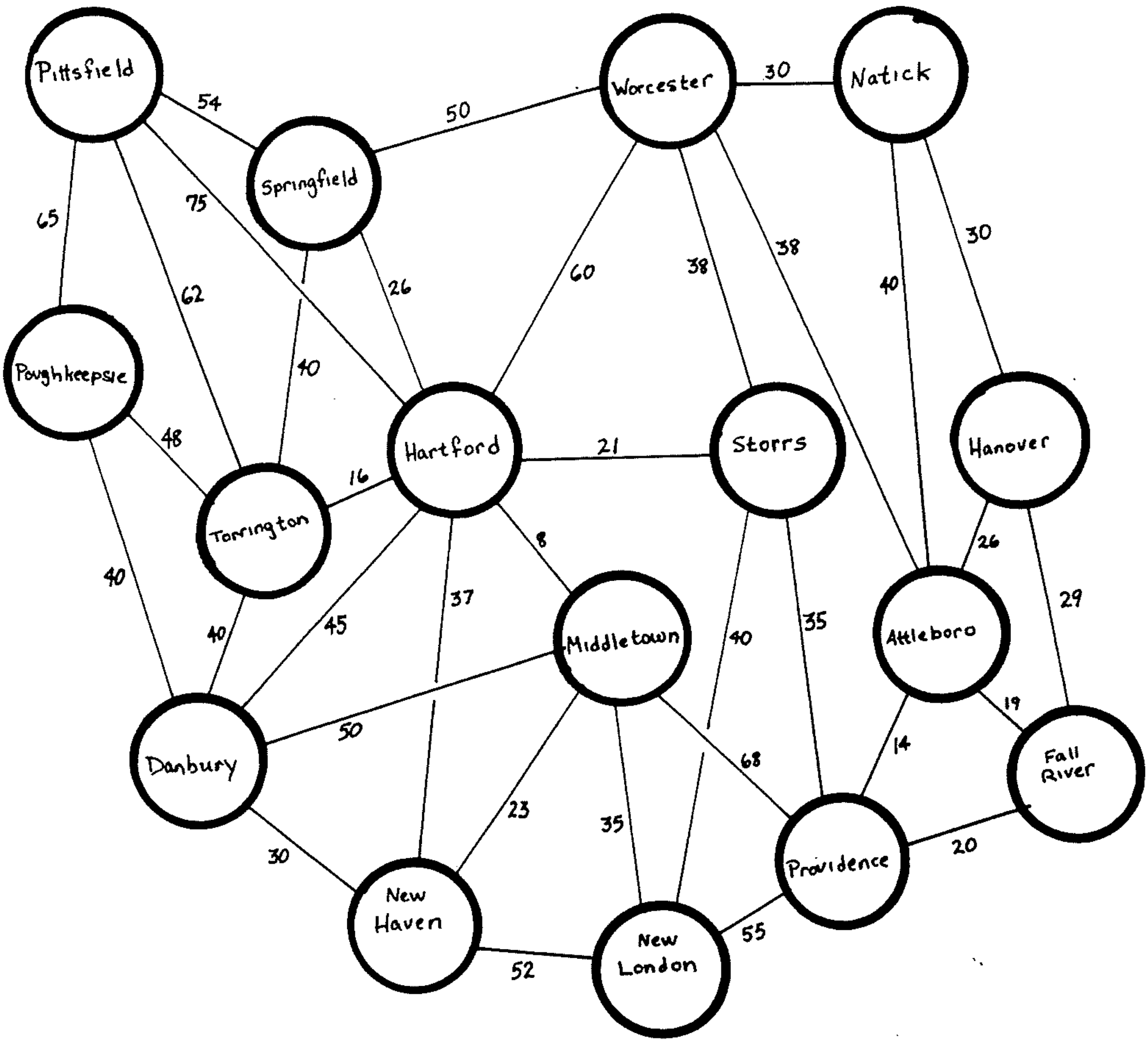


a. two years b. three years c. four years

[COZZENS, Johnson]

SHORTEST ROUTE ACTIVITY #4

Marvin Maybe wants to drive from Hanover, Massachusetts to Poughkeepsie, New York, to see his girlfriend at Vassar College. Use the map enclosed to find the shortest route. What is the length of the shortest route?

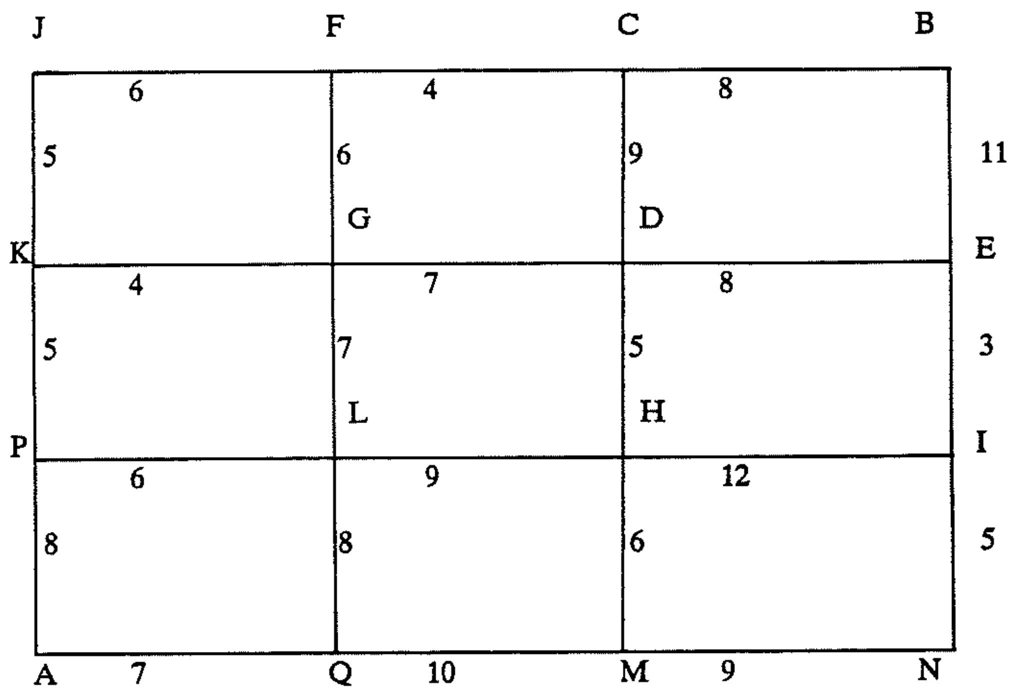


[COZZENS, Johnson]

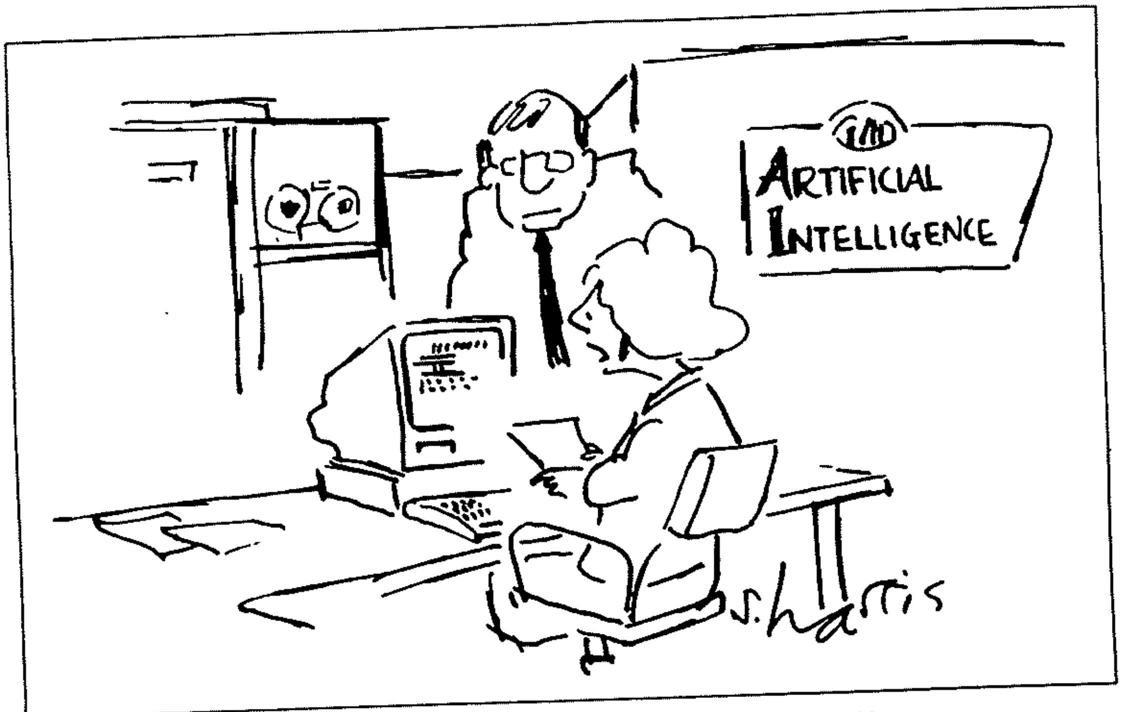
SHORTEST ROUTE ACTIVITY #5

A utility company must lay new power lines between a power substation A and a new school B. The lines must be laid in the streets. The cost (in thousands of dollars) of laying a line in each block is noted on the diagram below.

Find the path of least cost. Assume the starting position is substation A and the ending position is school B.



[Johnson]



"I gave it the traveling salesman problem. It said he should give up sales and go into banking."

SECTION VII: HAMILTONIAN CIRCUITS AND THE TRAVELING SALESMAN PROBLEM

TEACHING NOTES: INTRODUCTION

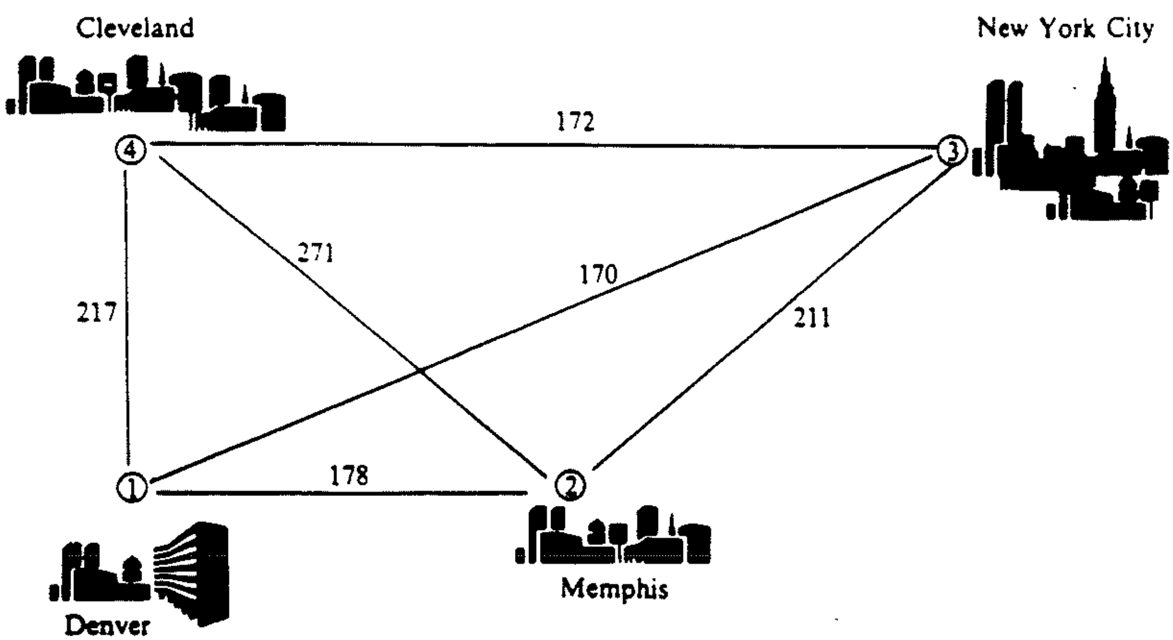
A businessman in Denver is planning a trip to visit clients in Memphis, Cleveland, and New York, before returning to Denver. The cost in dollars of traveling from one city to another is given in the diagram. The problem is to find the order in which these cities should be visited so that the total transportation cost is as small as possible.

The trip for the businessman is the same as a circuit since it starts at vertex 1 (Denver), visits each other vertex exactly once, and returns to vertex 1.

A circuit that passes through each vertex exactly once and begins and ends at the same vertex is called a Hamiltonian circuit.

So the businessman's problem becomes that of: Given a weighted graph, find a Hamiltonian circuit of least total weight. This is called the traveling salesman problem.

One way for the businessman to plan the trip is to always choose from the cities yet to be visited, the one that costs the least to get to. This strategy is called the nearest neighbor rule.

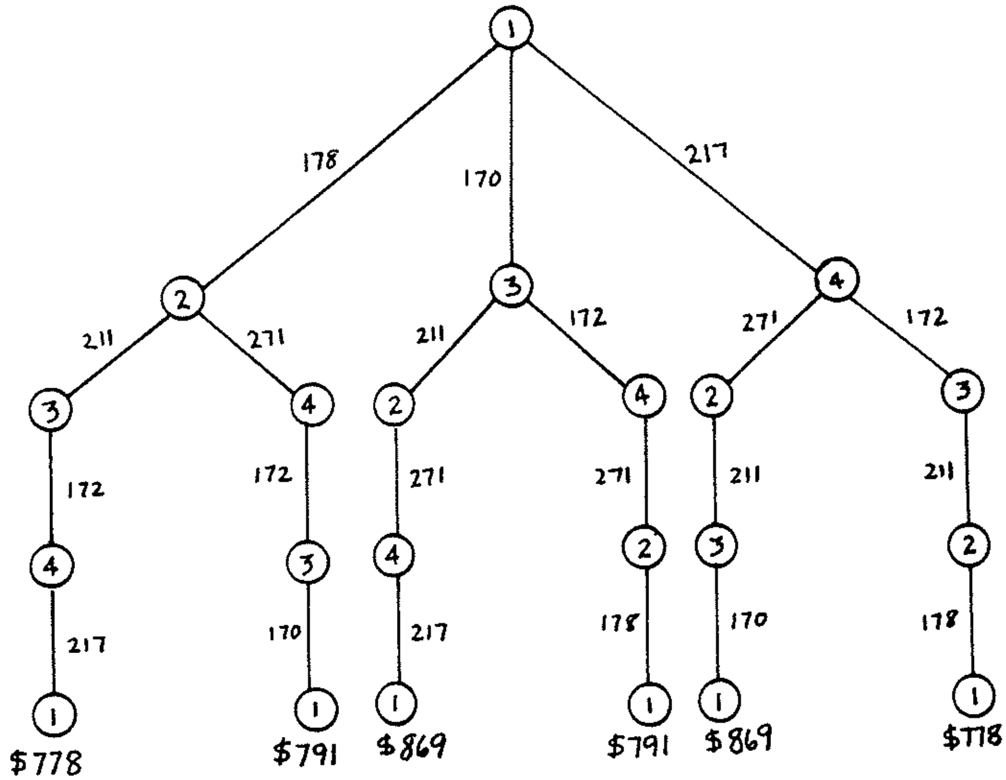


By listing all possible circuits and their costs you can see the Hamiltonian circuit with the minimum weight. To accomplish this, break the process of choosing a Hamiltonian circuit into steps. The first step is to pick the first vertex (the city to be visited). The second step is to pick the second vertex, and so on.

[Cozzens,Reynolds/Zangari]

TEACHING NOTES: SPANNING TREE FOR THE TRAVELING SALESMAN PROBLEM

The following diagram represents all possible spanning trees for the traveling salesman problem.



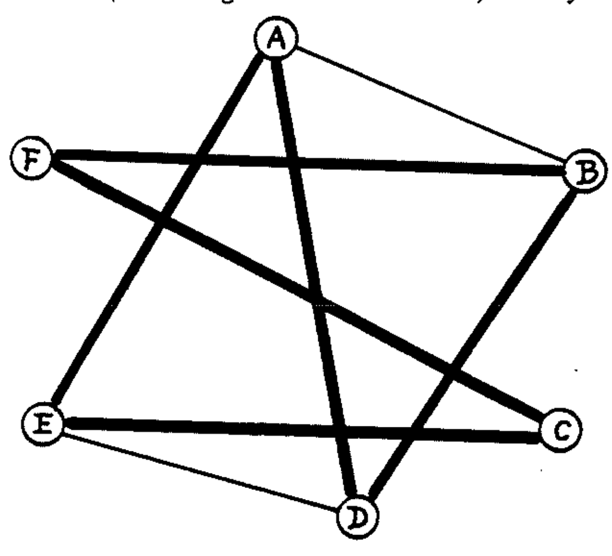
TEACHING NOTES: TRAVELING SALESMAN PROBLEM

Each path from the top to the bottom of the tree corresponds to a trip from Denver that visits each of the cities exactly once and returns to Denver. Adding the numbers on the edges along a path gives the cost of the circuit as listed in the bottom row. A circuit given by the nearest neighbor rule need not be a Hamiltonian circuit of least weight. Finding all possible tours, and picking the cheapest tour is not computationally efficient, since the number of tours is $[(n-1)!]/2$, which grows rapidly with n.

No. of vertices	*Time to List Circuits*
10	*less than 1 second*
15	*1 minute, 27 seconds*
16	*21 minutes, 48 seconds*
17	*5 hours, 49 minutes*
18	*4 days*
19	*2 months, 16 days*
20	*3 years, 10 months*
25	*more than 19 million years*

This table gives approximations of the time required to list by the computer all the Hamiltonian circuits in a graph in which each pair of vertices is joined by an edge when the computer can compute 1 billion computations per second. All Hamiltonian circuits quickly become impractical as the number of vertices in the graph increases. Unlike the Eulerian circuit problem, there is no known fast algorithm to solve the problem of finding a Hamiltonian circuit of least total weight. Since there is no known practical algorithm for the traveling salesman problem, it is natural to consider methods that are easy to implement, but do not always produce a Hamiltonian circuit of least weight. The nearest neighbor rule is only one such method.

The traveling salesman problem is NP-hard. This means that it is very unlikely that there is an efficient algorithm to solve the TSP (Traveling Salesman Problem) exactly on all graphs.



A - E - C - F - B - D - A

Chapter 2: STEINER POINTS

STEINER POINTS
TABLE OF CONTENTS

References and historical notes:

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Boys, C.V., Soap Bubbles and the forces which mould them, Doubleday Anchor, 1959

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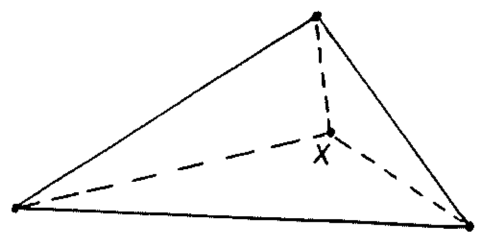
History: see Appendix C.

Construction of plexiglass models: see Appendix D.

*TEACHING NOTES: **Introduction/Overview***

The topic of "Steiner Points" resulted in an outpouring of creativity from many institute participants. In its simplest case, the problem is to find a point X inside a triangle for which the combined distances to the three vertices (the dotted lines in the diagram) is as small as possible.



The question of how to find the point X can be reformulated in a number of different mathematical contexts:

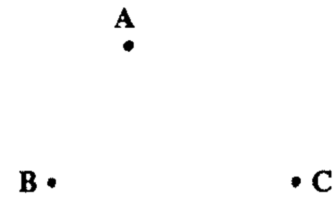
- *find a geometric construction for X.*
- *find a proof for the location of X.*
- *given coordinates for the vertices, find the coordinates of X experimentally, using analytic geometry, trigonometry, and/or calculators.*
- *experiment with this problem on the Geometric Supposer.*
- *generate an algorithm or a computer program which will find X.*
- *express the location of X as a function and use max/min techniques to find X.*

All of these perspectives are represented in this chapter. But there is more - involving applications to real-world problems and generalizations from these to four and even five given points. And, leaving the best for last, there are demonstrations involving soap bubbles, where students can discover how nature finds the point X by seeking the smallest soap film that will connect the points. Have fun!

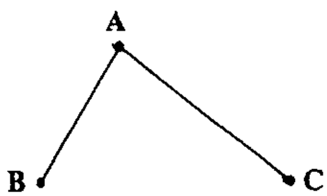
Joseph G Rosenstein

STEINER ACTIVITY #1

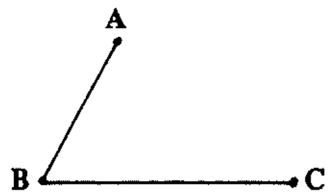
A Steiner problem asks for the shortest network of line segments that will interconnect a set of giver points, such as A, B, and C.



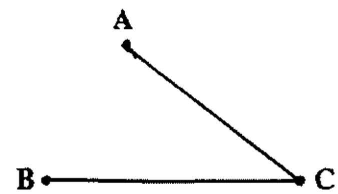
The possible networks that could be drawn are:



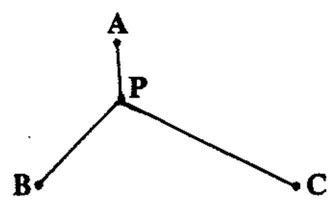
and



and



but none of these will be the shortest network, since one could find a fourth point, P, such that



AP + CP + BP provides a shorter network than any previous attempt.

Point P is a <u>Steiner Point</u> when the shortest network is determined, and its location is the subject of this unit.

Exercise: Measure each of the following to verify the claim above.

AB + AC = ___________

AB + BC = ___________

AC + BC = ___________

AP + CP + BP = _______

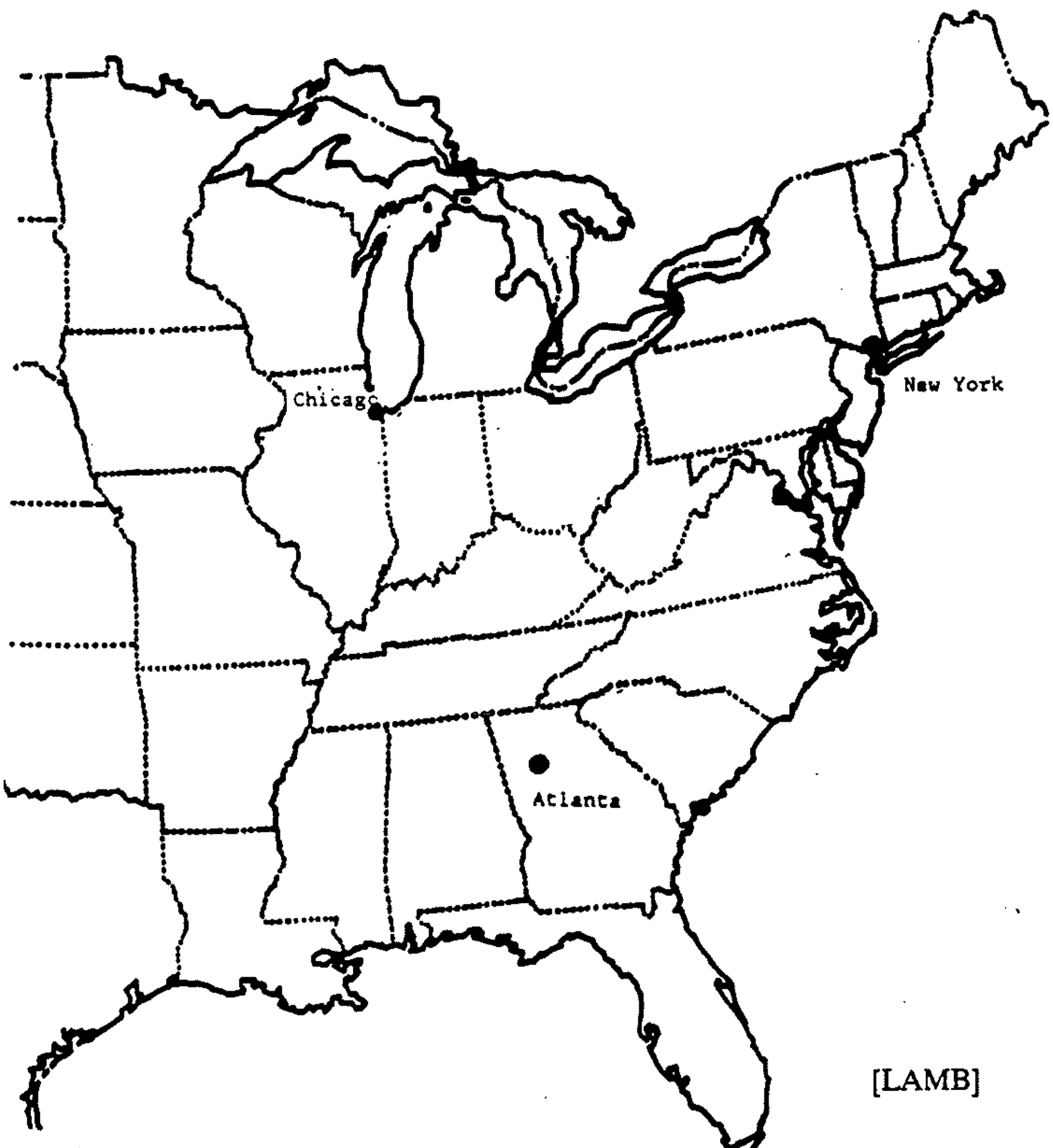
[Reynolds, Zangari, Van Hise]

STEINER ACTIVITY #2
Delta Airlines: Pricing Policy, and locating the Steiner Point

This is a real-life problem that arose in 1960 when it is told that Delta Airlines claimed that Bell Telephone overcharged them for the private lines between their main terminals in Chicago, New York, and Atlanta. The mathematicians at Bell Labs researched Delta's claims and found in the theory developed in the late 1800's by German mathematician Jacob Steiner that Delta was correct: A fourth point could be located so that the shortest route between the cities is a minimum, and this route could be the basis for the phone line charges.

Today Bell is not really concerned with the location of Steiner points, but in the length of the network that results if Steiner points are used. The cost of the service is based on the total length of the network.

Use the map below, and try to locate the position of the Steiner point for the Delta problem. Measure your path, and be sure it is a minimum.



[LAMB]

STEINER ACTIVITY #3

Elementary Soap Bubbles: Use soap solution and plexiglass models to illustrate existence of Steiner points.

Soap films show nature's solution to minimal path problems. Because of surface tension, a soap film is stable only if its area is a minimum. By experimenting with soap film, then, solutions can be found to problems which are extremely difficult to solve mathematically. Direct solutions would require the calculus of variations.

Soap Solution:

Use dish pan half full of water. On surface use DAWN dish soap and squirt one trail around edge of water. Be careful not to use too much soap (you can always add more soap). Then add two cap-fulls of glycerine to the solution. The soap will sit on the bottom of the dish pan. Gently swish the liquid with your hand, but do not create bubbles. This then is adequate to use for dipping the models.

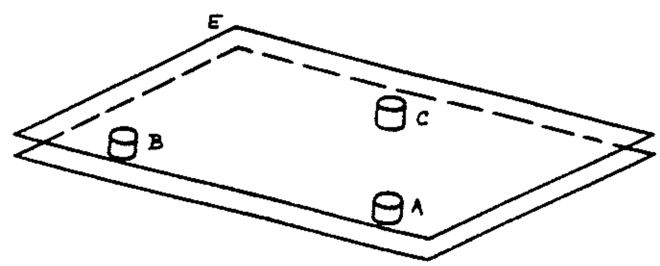
Plastic Models:

See appendix for instructions on construction.

Some basic shapes: Several triangles. Several quadrilaterals. A set of five vertices. Three dimensional models are particularly interesting, and have surprising results - see Boys and Perl for suggestions.

Dip the model gently and lift it from the solution at an angle slightly off the vertical. If you are holding the model at corner E, then the soap film will consist of line segments AB and AC.

model for triangle:



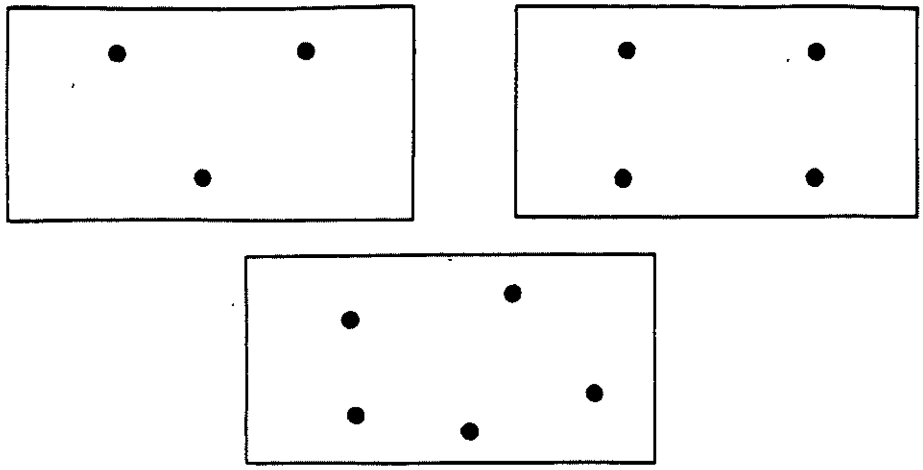
Class Activities:

Have the students guess the location of the path that the soap film will take, prior to dipping the model. Place the model on an overhead projector so that students can see clearly where the soap film has formed. Use a straw to blow very gently on the edge AC of the film near A so that a Steiner point will form. Have a template with 120 degrees to show that the Steiner point formed has three edges which meet at 120 degree angles. Superimpose this template on the plastic models after the film forms, and after students have guessed the angle measure.

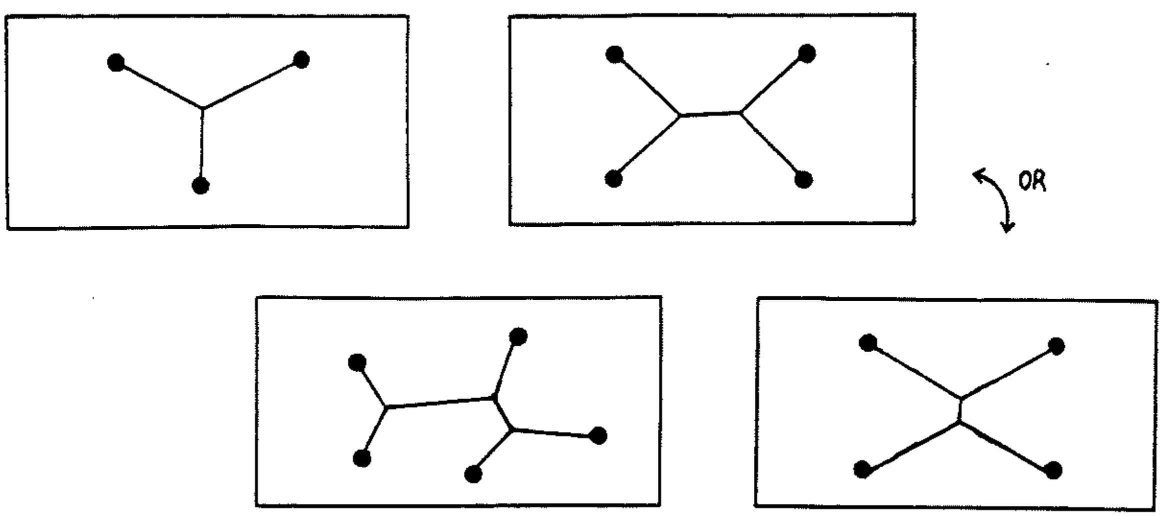
With models which have more than three vertices, there may be more than one Steiner Point; you and your class can experiment with forming soap films and blowing them at various places.

[LAMB, PERL]

GUESS THE SHAPE THESE PATTERNS WILL PRODUCE:



Solutions to two-dimensional problems:



Steiner Activities #6 and #7 return to the problem of computing Steiner Points using soap bubbles.

STEINER ACTIVITY #4

Geometry Supposer: Speculate on the location of the Steiner Point

Software required: Geometry Supposer --- Circles by Sunburst

1) Press (N) new shape.
2) Press (1) one circle.
3) Press (2) your own.
4) Choose a radius of 3 or 4 units.
5) Press (2) label, then press (5) moveable point, then press (1) straight lines (choose 2 points outside of the circle, B, C, so that all together, A, B, and C will form a triangle).
6) Press Esc twice, then press (3) erase, then press (1) erase circle A.
7) Press Esc, then press (1) draw, then press (1) segment AB, press segment AC, press (1) again segment BC, then press Esc.

At this point, you should have a triangle ABC. The following steps will construct the Steiner Point. First, draw equilateral triangle ACE.

8) Press (4) circle, then press (2) center and radius of the circle, use center A , radius of AC * 1.
9) Press (4) circle, then press (2) center and radius, center C, radius of AC * 1.
10) Press Esc, then press (2) label, then press (1) intersection, then press (1) two circles A (space bar then return key) C.

Erase construction lines.

11) Press Esc, then press (3) erase, then press (1) circle C, press (1) erase circle A, press space bar and the return key.

Circumscribe triangle AEC.

12) Press Esc, then press (1) draw, then press (4) circle, then press (1) points on the circumference AEC.

Now find the Steiner Point, the intersection of BE with the circle.

13) Press (1) segment BE.
14) Press Esc, then press (2) label, then press (1) intersection, then press (2) circle and line, circle F and line BE. (G is the Steiner point).
15) Press Esc, then press M measure, press (3) angle AGB, press return, press (3) angle CGB, press return, press (3) angle CGA. They should all equal 120 degrees, or 2.09 radians.
16) Press Esc, then press (1) draw, press (1) segment AG, press (1) segment BG, press (1) segment CG.

[Zangari]

Exercise

1) Measure the length of AG, BG, CG, and BE. Do you see a relationship? Form a conjecture.

2) Repeat the algorithm letting the 2 moveable points form a right triangle. Repeat answering Question #1.

3) Repeat the algorithm letting the 2 moveable points form an obtuse triangle whose obtuse angle is greater than 120 degrees. Repeat answering Question #1.

TEACHING NOTES: Steiner Constructions

Topic: *Steiner Points Constructions for three, four, and five points, using compass and straightedge. These constructions provide an excellent integrated review and application of the basic constructions listed below.*

Prerequisite skills:

Basic constructions, including:

- *copy a line segment*
- *perpendicular bisector of a segment*
- *bisector of an angle*
- *construct a triangle given 3 sides*
- *construct an equilateral triangle*
- *locate the circumcenter of a triangle*
- *construct the circumcircle of a triangle*

Warning: Use caution in placing more than 3 points in all Steiner point constructions - it doesn't take long for the construction to "fall off" the page.

I. *3-POINT Construction:*

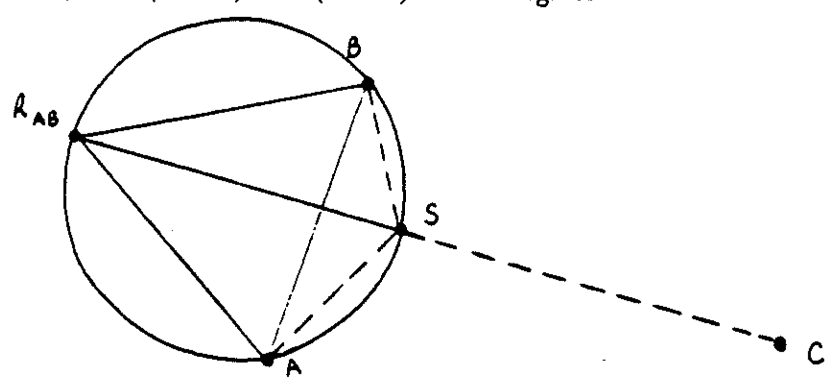
Given any three points A, B, C.

Arbitrarily choose any 2 of the 3 points, say A and B, and construct their replacement point R_{AB} as follows:

1. *Construct an equilateral triangle, ABR_{AB}, using AB as one side*
2. *Circumscribe a circle about ABR_{AB}*
3. *Draw segment $R_{AB}C$*
4. *Point S, where $R_{AB}C$ intersects arc AB is the Steiner Point, that is,*

   $AS + BC + CS$ is minimal

   $m(\angle ASC) = m(\angle CSB) = m(\angle BSA) = 120$ degrees



Point R is the replacement point for points A and B in the sense that it replaces both A and B in the search for a minimal spanning tree. Any replacement point will be designated by a subscript which names the two points it replaces. So point R is denoted R_{AB}. It is proved elsewhere that

$$R_{AB}C = AS + BS + CS.$$

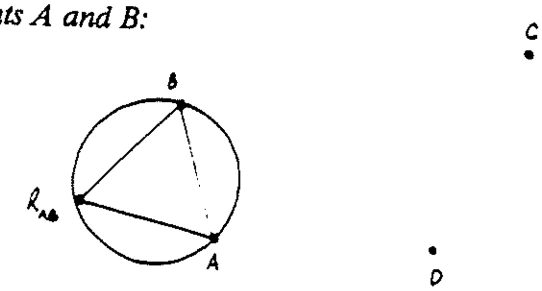
This means that the "network" joining R_{AB} and C is the same size as the minimal network joining A, B, and C; thus R_{AB} does "replace" A and B in the network of points A, B, and C. For three points, the Steiner Point is unique and could have been found using either BC or AC as the side upon which to construct the equilateral triangle. [Eldi, Schumacher]

Note: Only Section I is needed for Steiner Activity #5, which involves construction of Steiner Points for triangles.

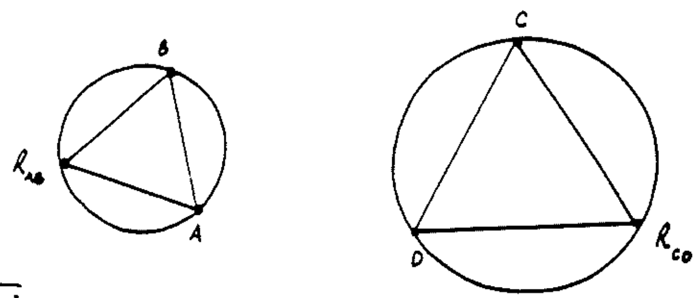
II. 4-POINT Construction:

Given A, B, C, and D;
The technique will be to find one replacement point for A and B, and then another for C and D.

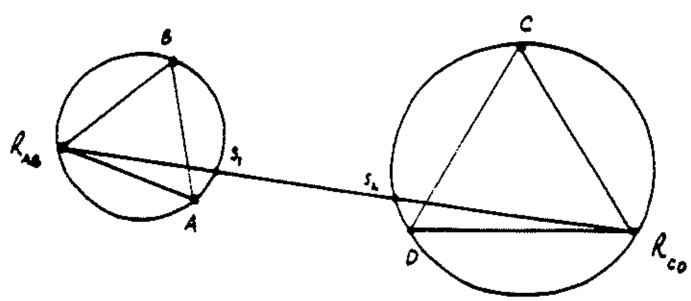
1. *Construct R_{AB} for points A and B:*



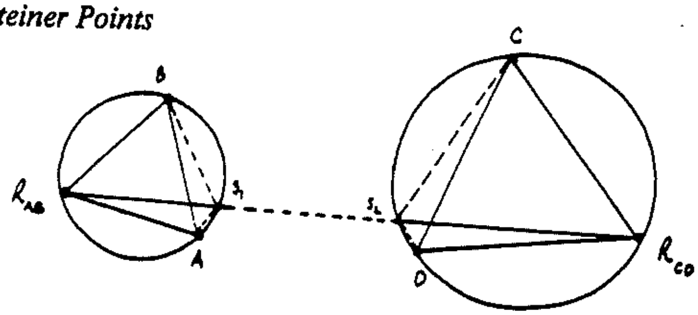
2. *Construct R_{CD} for points C and D:*



3. *Draw segment $\overline{R_{AB}R_{CD}}$:*



4. *S_1 and S_2 are two Steiner Points*



The network of dotted lines is a minimal network joining the four points A, B, C, and D.

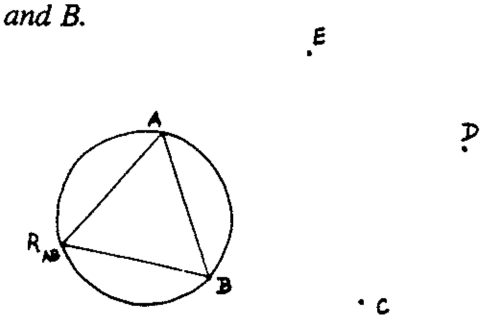
In constructing 2 Steiner Points for a 4-point network, it should be noted that the solution is not unique. In fact, another set of points could have been located by constructing R_{BC} and R_{AD}.

III. ***5-POINT Construction:***

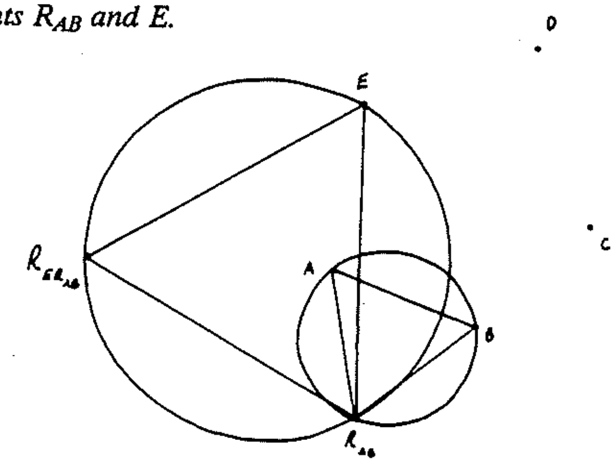
Given A, B, C, D, and E.

The technique will be to use a replacement point for 2 of the points, reducing the network to 4 points. Then, the same strategy used in the previous section will be used.

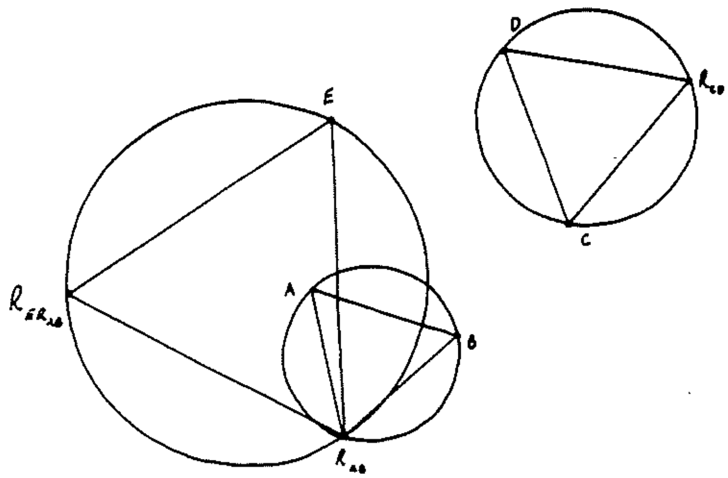
1. *Construct R_{AB} for points A and B.*



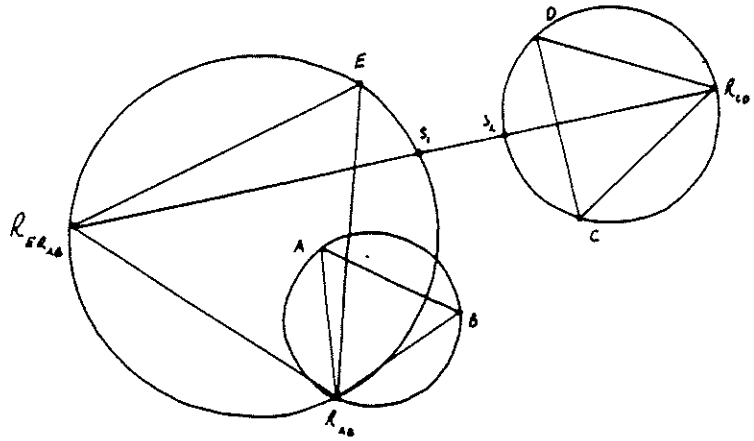
2. *Construct $R_{ER_{AB}}$ for points R_{AB} and E.*



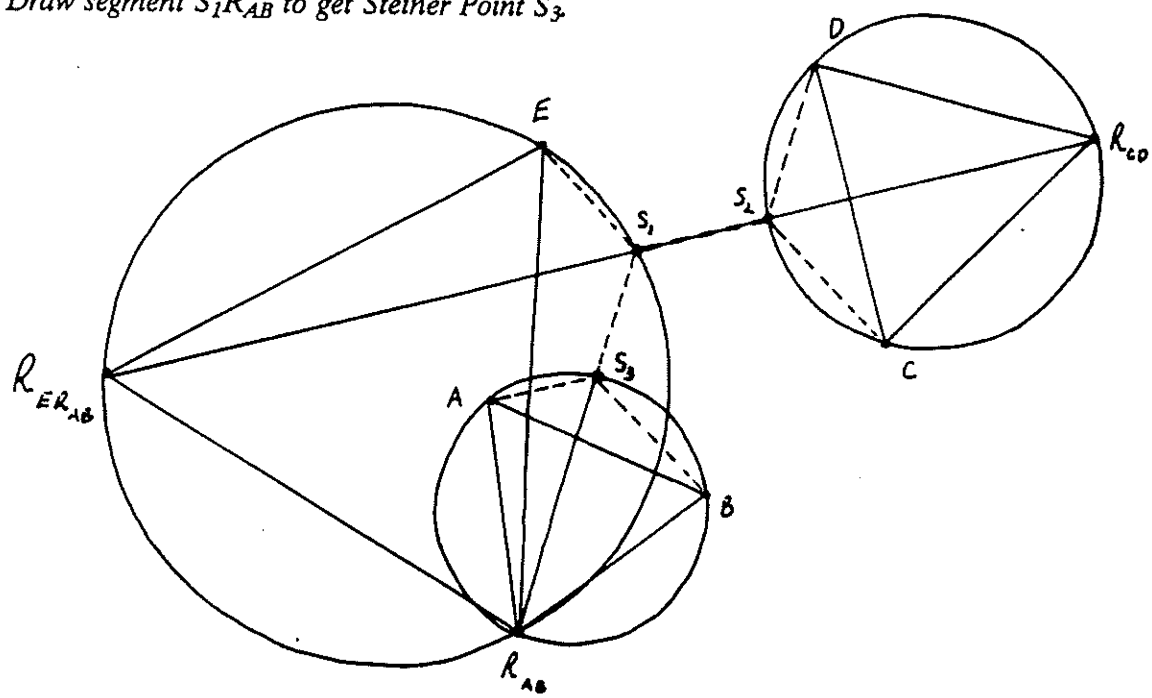
3. *Construct R_{CD} for points C and D.*



4. *Draw segment $\overline{R_{ER_{AB}}R_{CD}}$ establishing Steiner Points S_1 and S_2.*



5. *Draw segment $\overline{S_1R_{AB}}$ to get Steiner Point S_3.*



The network of dotted lines is a minimal network joining points A, B, C, D, and E.

This construction can be done in ten different ways, since we can choose any one of five edges to begin and then any two opposite edges to carry out the 4-point construction. Each attempt does not necessarily yield a minimal spanning tree - sometimes contradicting constraints on the points arise during the construction.

IV. *Computational Complexity*

At each step in the process, replacement points are found for two given points, a given point and a previous replacement point, or two previous replacement points. Also, each equilateral triangle can be oriented on either side of the line segment. The algorithm used, known as Melzak's algorithm, then determines the Steiner Points for each replacement point as it 'backs out'. To determine the best of the 'minimal' spanning trees found, all possible trees would have to be examined. For n points, the points would have to be combined two at a time until 3 points were left, at which time only one Steiner Point could be found. However, a computer program would still try $_3C_2$ or 3 ways to find the unique point. The number of ways of choosing the points two at a time and reducing the network to two points is

$$({}_nC_2)({}_{n-1}C_2)({}_{n-2}C_2)\ldots({}_4C_2)({}_3C_2),$$

where $_nC_2$ is the number of ways of choosing two elements from an n element set. Also, the number of triangles drawn is n-2, so the number of different ways of drawing them is 2^{n-2} (2 orientations each). So, a conjecture for the number of possible spanning trees to be examined in a worst-case scenario of Melzak's algorithm is:

$$2^{n-2}({}_nC_2)({}_{n-1}C_2)({}_{n-2}C_2)\ldots({}_4C_2)({}_3C_2)$$

This conjecture is an example of "combinatorial explosion", a growth rate that dwarfs the worst polynomial time. The following chart shows the explosion:

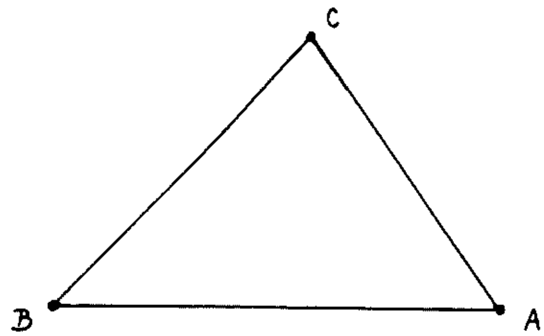
n	*number of cases*
3	*6*
4	*72*
5	*1,440*
6	*43,200*
7	*1,814,400*

Although methods have been developed to cut down on the number of cases a computer would have to check, it is still an enormous task. Even with refinements, the current computer limit is 29 points. For more on the subject of the combinatorial explosion and Melzak's Algorithm, see "The Shortest Network Problem" in Scientific American, January 1989, by Marshall Bern and Ron Graham.

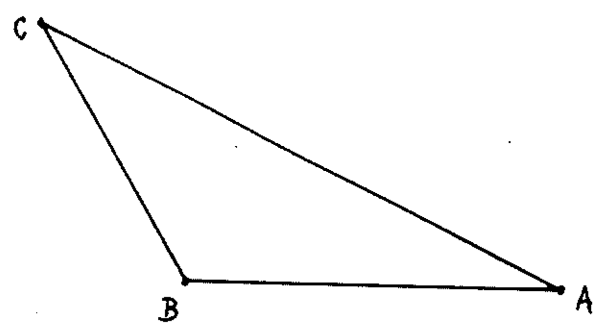
[Eldi, Schumacher]

Steiner Activity #5
Steiner Constructions: Special Triangles

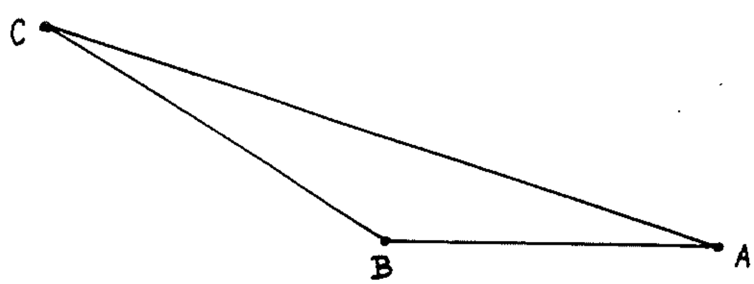
1. Find the Steiner Point for the three given points which form an acute triangle.



2. Construct the Steiner Point for the three given points which form an obtuse triangle with a 120 degree angle.



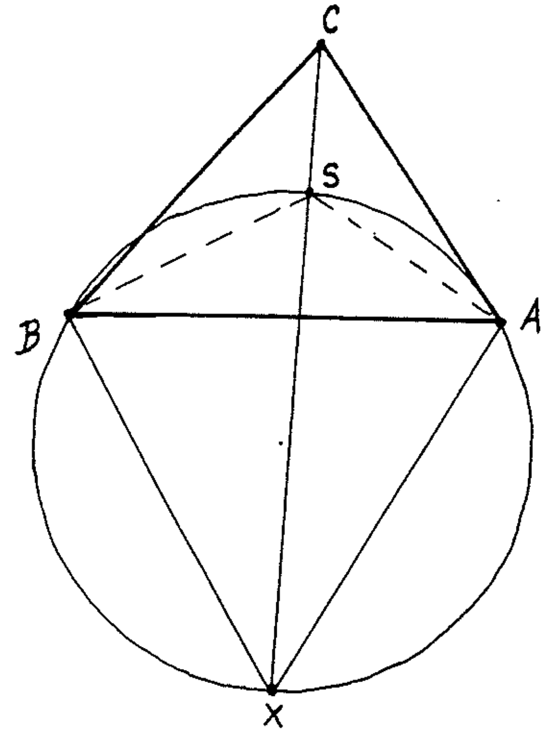
3. Construct the Steiner Point for the three given points which form an obtuse triangle with an angle of 150 degrees.



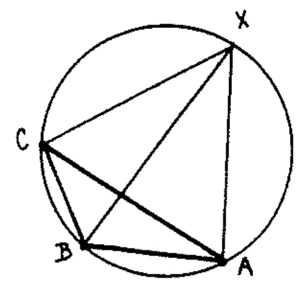
[Zangari, Reynolds]

Steiner Activity #5 - Answers

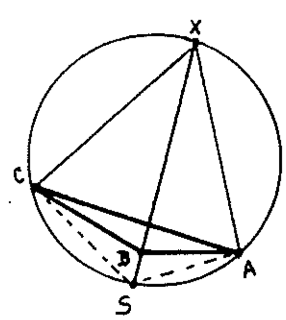
1. Acute triangle:



2. 120 degree angle:
   Vertex B is the Steiner Point. The Steiner Point is a vertex, and so the Steiner Network is the same as the sides.



3. 150 degree angle:
   The Steiner Point is outside, and the network formed is not a minimum.

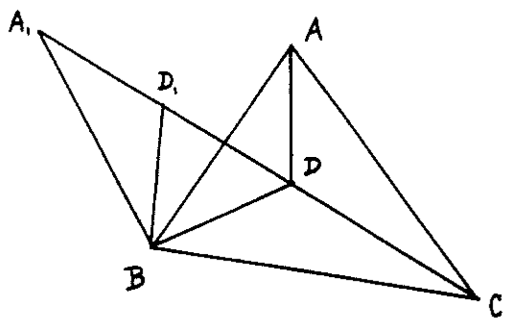


TEACHING NOTES: Steiner Proofs

Here follow proofs of two important theorems which justify the Steiner Point construction and prove that the network is the shortest one possible. These proofs are included as teaching notes, but could be used as activities in a geometry class. Part III contains a discussion by Eldi and Schumacher on obtuse triangles.

I. <u>Proof of Steiner's Theorem</u> that the degrees of the angles at the Steiner point is 120 degrees [LAMB]

<u>Theorem</u> 1: If points A, B, and C form a triangle so that no angle of the triangle exceeds 120 degrees, and if D is a point in the interior of triangle ABC so that the sum AD + BD + CD is a minimum, then the $m(\angle ADC) = m(\angle ADB) = m(\angle BDC) = 120$ degrees.



<u>Construction:</u> First rotate triangle ABD 60 degrees counterclockwise around point B. That is, segment $\overline{BA}$ rotates to $\overline{BA_1}$. $\overline{BD}$ rotates to $\overline{BD_1}$, and $\overline{AD}$ rotates to $\overline{A_1D_1}$ so that $m(\angle A_1BA) = m(\angle D_1BD) = 60$ degrees.

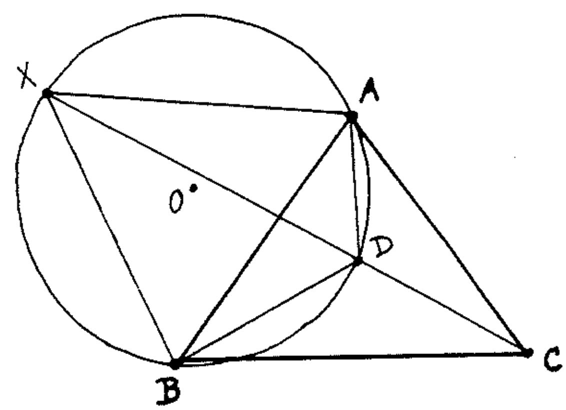
<u>STATEMENTS</u>	*<u>REASONS</u>*
1. $\triangle A_1BD_1 = \triangle ABD$	1. Rotation preserves congruency of triangles.
2. $\overline{A_1D_1} = \overline{AD}$, $\overline{BD_1} = \overline{BD}$	2. Rotation preserves congruency of segments.
3. $\angle BD_1D = \angle BDD_1$	3. Rotation preserves congruency of angles.
4. $m\angle D_1DB = 60°$	4. Definition of the rotation's magnigude.
5. $m\angle BD_1D = m\angle BDD_1 = 60°$	5. Rotation preserves angle measure.
6. $\triangle DBD_1$ is equilateral	6. Definition of an equilateral triangle.
7. $\overline{BD} \cong \overline{DD_1}$	7. Definition of an equilateral triangle.
8. $AD + BD + CD = A_1D_1 + D_1D + CD$	8. Substitution.
9. $A_1D_1 + D_1D + CD$ will be a minimum when A_1, D_1, D, and C are collinear.	9. Triangle Inequality.
10. $\angle D_1DB$ and $\angle BDC$ are supplementary $\angle A_1D_1B$ and $\angle BD_1D$ are supplementary	10. Definition of Supplementary Angles.
11. $m\angle BDC = m\angle A_1D_1B = 120°$	11. Definition of Supplementary Angles.
12. $m\angle ADB = 120°$	12. Supplementary Angle Theorem.
13 $m\angle ADC = 120°$	13. Supplementary Angle Theorem.

II. Proof to justify the Steiner Construction

Theorem 2: If D is the Steiner Point in triangle ABC, O is the circle circumscribing $\triangle ADB$, and CD intersects the circle at X, then $\triangle ABX$ is equilateral and

$$XC = AD + BD + CD.$$

(The orientation of the equilateral triangle ABX must be as in the diagram below.)



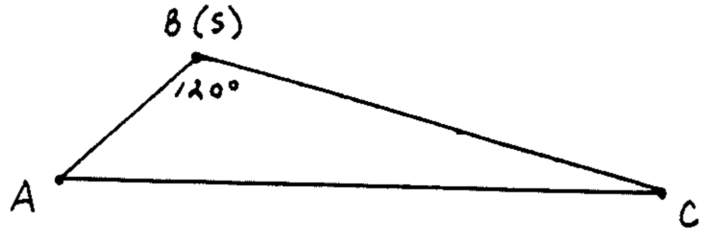
STATEMENTS	REASONS
1. $m(\angle ADB) = 120°$	1. Theorem 1.
2. $\therefore \widehat{AXB} = 240°$	2. Arc is double the angle.
3. $\therefore \widehat{ADB} = 120°$	3. Subtraction.
4. $m(\angle AXB) = 60°$	4. Arc is double the angle.
5. $m(\angle ADC) = 120°$, $m(\angle BDC) = 120°$	5. Theorem 1.
6. $m(\angle ADX) = 60°$, $m(\angle BDX) = 60°$	6. Supplementary angles.
7. $\widehat{AX} = 120°$, $\widehat{BX} = 120°$	7. Arc is double the angle.
8. $m(\angle ABX) = 60°$, $m(\angle BAX) = 60°$	8. Arc is double the angle.
9. ABX is equilateral triangle.	9. Definition of equilateral triangle.
10. $AD \cdot BX + BD \cdot AX = DX \cdot AB$	10. Ptolemy's Theorem.
11. $AB = BX = AX$	11. Definition of equilateral triangle.
12. $AD \cdot AB + BD \cdot AB = DX \cdot AB$	12. Substitution property.
13. $AD + BD = DX$	13. Division property of equality.
14. $XD + DC = CX$	14. Betweenness theorem.
15. $AD + BD + DC = XC$	15. Substitution of (13) into (14).

Ptolemy's Theorem: If a quadrilateral is inscribed in a circle, then the sum of the products of the lengths of the opposite sides is equal to the product of the diagonals of the quadrilateral.

III. Discussion of the Steiner Point in obtuse triangles with degree > = 120 degrees. A case study.

As was proven in the previous pages, the Steiner Point is inside the triangle for any triangle whose angles are each less than 120 degrees. If one angle of the triangle is exactly 120 degrees, the limiting position of the Steiner Point S coincides with the vertex B.

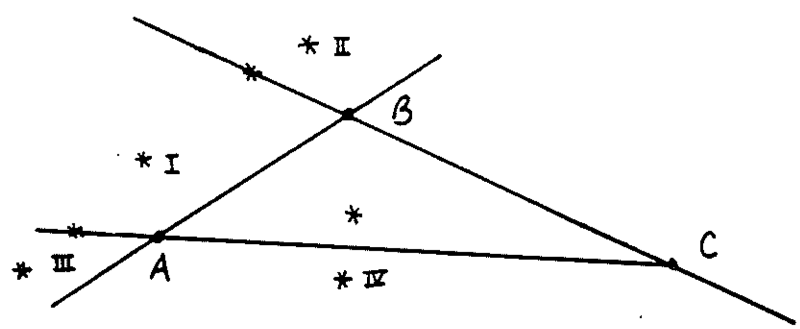
Diagram 1



$AS + BS + CS = AB + BC$

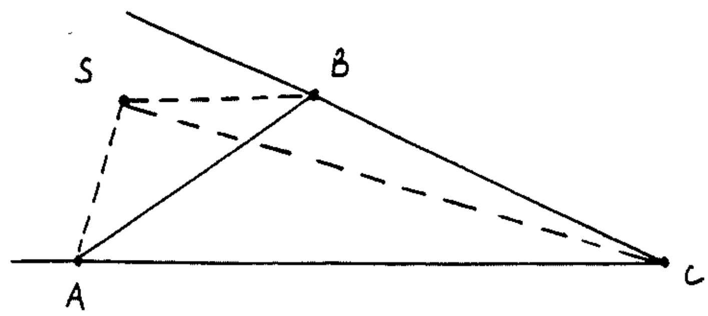
*Can the Steiner Point be outside the triangle if the measure of $\angle ABC > 120°$? If so, a look at diagram 2 below shows that considering it to be in regions I, II, III, IV or on sides $\overline{CB}$ or $\overline{CA}$ extended represents all possible locations (see *).*

Diagram 2



[Eldi, Schumacher]

Diagram 3



Case 1: See Diagram 3. Let S be any point in region I. In △ASB,

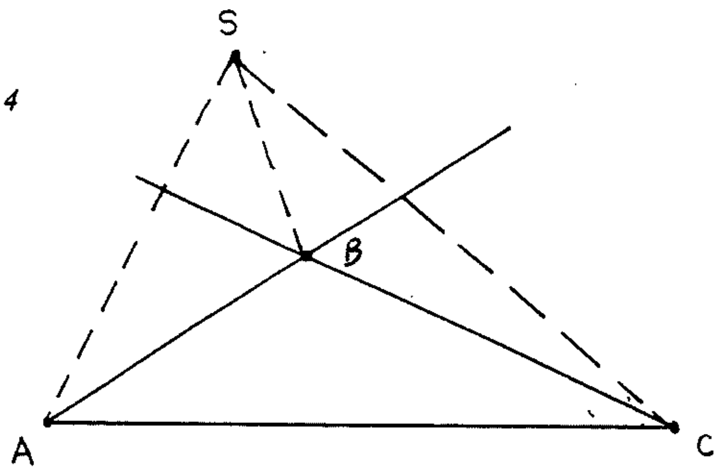
$$AS + BS > AB.$$

In △SBC, ∠SBC is obtuse, so

$$CS > BC.$$

Adding, AS + BS + CS > AB + BC, so S is not a minimum distance point since B is better.

Diagram 4



Case 2: See Diagram 4. Let S be any point in region II. Since m∠ABC < 180°, m(reflex∠ABC) > 180°, so ∠SBA or ∠SBC is obtuse (possibly both). Assume ∠SBC is obtuse. Since ∠SBC is obtuse,

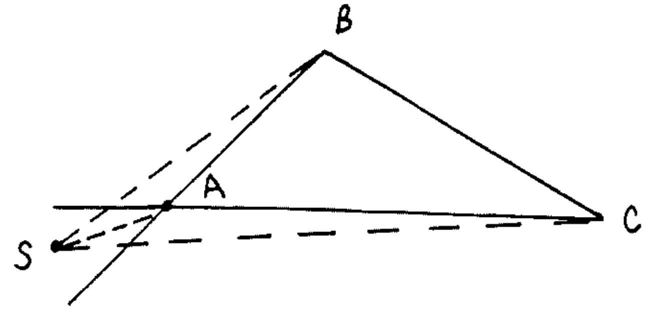
$$CS > BC.$$

In △SBA,

$$AS + BS > AB.$$

Adding, AS + BS + CS > AB + BC again.

Diagram 5



Case 3: See Diagram 5. Let S be any point in region III. Since $\angle SAB$ is obtuse,

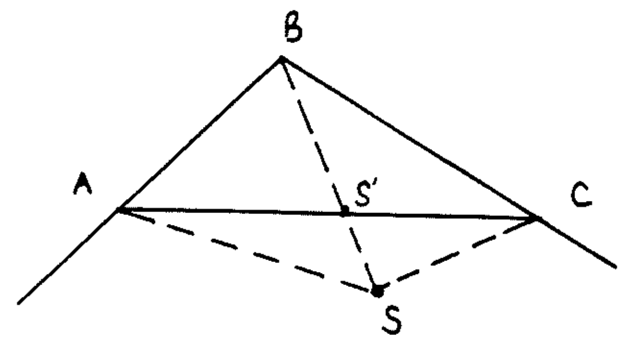
$$BS > AB.$$

In ΔSAC,

$$AS + CS > AC > BC.$$

Adding, $AS + BS + CS > AB + BC$.

Diagram 6



Case 4: See Diagram 6. Let S be a point in region IV. Note that

$$AS + SC > AC,$$

and if S is replaced by S', where $\overline{BS}$ intersects $\overline{AC}$, we get a shorter network:

$$AS + SC > AC,$$

and

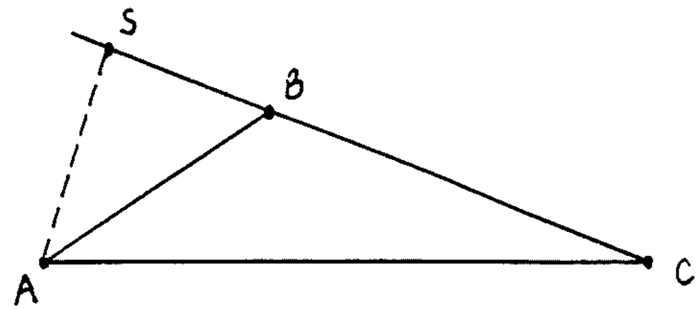
$$BS > BS',$$

so

$$AS' + CS' + BS' < AS + BS + CS.$$

This shows that AS + BS + CS is not the length of the shortest network, by showing that there is a shorter network. (Note that the shorter network constructed is not the shortest one, it just demonstrates that the original network in region IV is not the shortest one).

Diagram 7



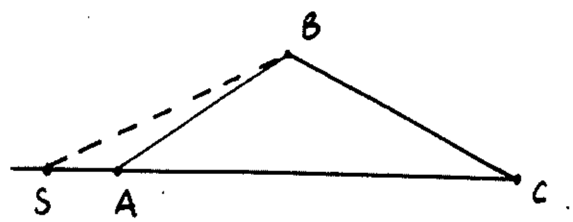
Case 5: See Diagram 7. Let S be a point on $\overline{CB}$ extended. In $\triangle SAB$,

$$AS + BS > AB.$$

Also, CS > BC. Adding, we again obtain

$$AS + BS + CS > AB + BC.$$

Diagram 8



Case 6. See Diagram 8. Let S be a point on $\overline{CA}$ extended. In $\triangle SAB$,

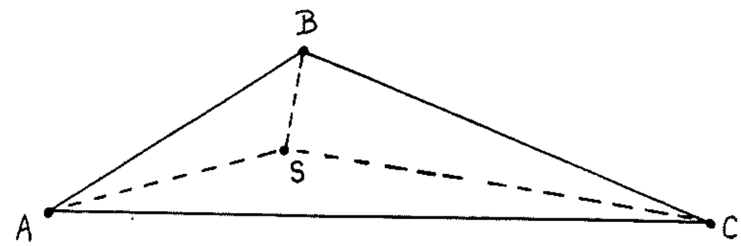
$$AS + BS > AB.$$

Also, CS > AC > BC. Then

$$AS + BS + CS > AB + BC.$$

Therefore, for an obtuse triangle with an obtuse angle of measure equal to or greater than 120 degrees, the Steiner Point is at the vertex of the obtuse angle.

Diagram 9



Case 7: See Diagram 9. Let S be a point inside the triangle (or on segment AC). Observe that in Theorem 1, the hypothesis that no angle exceeds 120 degrees is never used. But if m(∠ABC) > 120 degrees, then m(∠ASC) > m(∠ABC) > 120 degrees, contradicting Theorem 1.

TEACHING NOTES: Precalculus, analytic geometry

Most analytic geometry students lack the prerequisite skills to deal with maximum/minimum problems that are easily solved by calculus techniques. However, that does not mean they cannot deal with this topic from an intuitive or inductive reasoning standpoint. The exercises in this section have students calculating minimum network distances to three points from many "center" points for the purpose of leading students to see what center point seems to yield the minimum. From using several isosceles triangles, it is hoped that students can generalize their results to predict the location of the Steiner Point, the point where the network distance will be a minimum.

The soap bubble experiments can test the students predictions, first with isosceles triangles and later for acute scalene triangles, and still later for obtuse scalene triangles.

[Reynolds]

STEINER ACTIVITY #6

Exercises for Analytic Geometry: Computing minimum network distances from many "center" points.

A. For ease of computation, let the coordinates of A, B, and C be (-18,0), (18,0), and (0,24). Note that all of the "M" points will lie on the y-axis.

Use the methods of analytic geometry to compute

1. The centroid (the concurrence of medians).
2. The incenter (concurrence of angle bisectors).
3. The circumcenter (concurrence of perpendicular bisectors of sides).
4. The orthocenter (concurrence of altitudes).

B. Use geometrical methods to determine M (on the altitude from C to AB) where the measures of angle MAB and MBA are each:

1. 30 degrees
2. 45 degrees

C. For each M, determine the "inside" distance: AM + BM + CM.
Use a calculator to compute these distances to two decimal places.

D. Fill in numbers 1 - 7 which compare all the distances:

1. Shortest "outside" distance __________

Inside distances using:

2. Centroid __________
3. Incenter __________
4. Circumcenter __________
5. Orthocenter __________
6. m(angle MAB) = 30 degrees __________
7. m(angle MAB) = 45 degrees __________

E. Which distance seems to be the shortest? __________

[Reynolds]

F. Redo parts A - D using points A (-16,0), B (16,0). amd C (0,30).

1. Shortest "outside" distance ____________

Inside distances using:

2. Centroid ____________
3. Incenter ____________
4. Circumcenter ____________
5. Orthocenter ____________
6. m(angle MAB) = 30 degrees ____________
7. m(angle MAB) = 45 degrees ____________

G. Add m(angle MAB) = (integral degree angles between 20 and 40). Use trigonometric methods to compute the "inside" distances.

H. Based upon your computations in D, F, and G, how would you locate the "center" for shortest "inside" distance if given an isosceles triangle?

____

____

____

[Zangari, Reynolds]

STEINER ACTIVITY #6 - ANSWERS

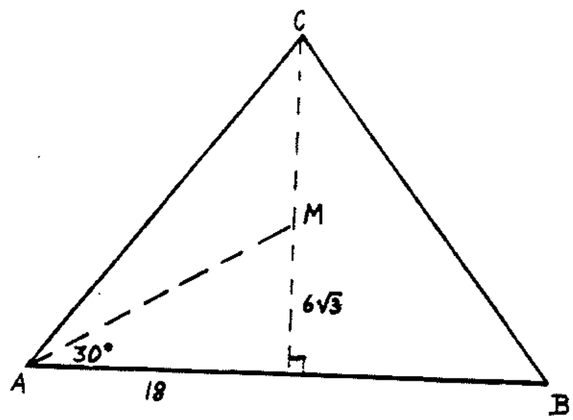
A) 1) midpoint CB, P(9,12), equation of median line AP: $y - 12 = 4/9(x - 9)$
midpoint AB, Q(0,0), equation of median line CQ: $x = 0$
centroid: (0,8)

2) equation of AC: $4x - 3y + 72 = 0$
equation of AB: $y = 0$
equation of angle bisector of angle CAB: $\frac{4x - 3y + 72}{5} = \frac{y}{1}$
which yields: $4x - 8y + 72 = 0$
equation of angle bisector of angle ACB: $x = 0$
incenter: (0,9)

3) equation of perpendicular bisector of BD: $x = 0$
circumcenter: (0,5.25)

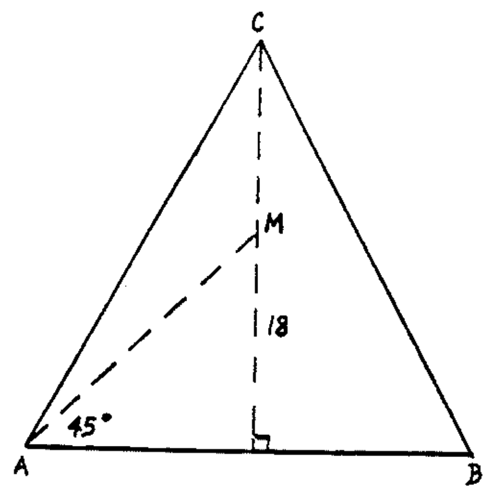
4) equation of altitude from A: $y = 3/4(x - 18)$

B) 1)



M(0,6 3)

2)



M(0,18)

D) 1. 60
2. 55.40
3. 55.24
4. 56.25
5. 55.50
6. 55.18
7. 56.91

E) 55.18 (angle measure of 30 degrees)

F) 1. 66
2. 57.74
3. 57.36
4. 57.49
5. 57.73
6. 57.71
7. 59.25

H) m(angle MAB) = 30 degrees

STEINER ACTIVITY #7
Precalculus: Discovering Steiner Point to optimize network

The objective of this activity is to test the predictions made in Activity 6 for locating the center for a minimum network of three points making an isosceles triangle and extending this prediction to any triangle. Using plexiglass models (see Appendix C) and soap bubbles, the Steiner Point or point for minimum network distance can be seen.

Exercise

1. Use several isosceles triangle models, preferrably in ratios to match calculations in Activity 6. They were based on two 3,4,5 or 8,15,17 triangles being placed side to side to make one isosceles triangle with integer height. (For example, in the first triangle, two 18,24,30 (3,4,5) triangles are put together to form 1 isosceles triangle with base 36, two 30 unit sides, and height 24).
2. Dip the models in the soapy water keeping one side of the triangle down while lifting the model from the soapy water. (A film of soap should form lines connecting the two sides not kept down.) Using a straw, blow gently at one side that has the film of soap. (The soap should move to an equilibrium point which should locate the Steiner Point). How does the Steiner Point compare with M (from Activity 6) for the shortest network distance?

   ____

   ____
3. Use soap bubbles to find the Steiner Points of several isosceles triangles. What feature(s) about the location of the Steiner Point (other than it is located on the altitude to the base) of an isosceles triangle seems consistent? That is, given an isosceles triangle, how would you locate the Steiner Point?____

   ____

   ____
4. Use soap bubbles to find the Steiner Points of several acute scalene triangles. Does your conclusion from number 3 hold for these triangles? Explain.

   ____

   ____

   ____
5. Use soap bubbles to find the Steiner Points of several scalene obtuse triangles with obtuse angles ranging from 100 degrees to 140 degrees. Which triangles have Steiner Points?

   ____

   ____

   Which do not?____

   ____
6. Does this seem consistent with your observations from numbers 3 and 4? Explain.

   ____

   ____

   ____

[Zangari, Reynolds]

Further activities with soap bubbles occur in Steiner Activities #9 and #10.

Answers - Activity 7

5) A Steiner Point will exist for all triangles which have all angles less than 120 degrees. At an angle of 120 degrees, the Steiner Point falls on one of the sides. For angles larger than 120 degrees, no Steiner Point will exist.

STEINER ACTIVITY #8
Analytic Geometry

1. For an equilateral triangle, find the minimum distance using the Steiner Point.
   a. If the length of a side is 6 feet.

   b. If the length of a side is s.

   c. What % shorter is the minimum distance than the sum of two sides?
      In (a)

      In (b)

2. For an isosceles triangle, find the minimum distance using the Steiner Point.
   a. If the base is 6 feet, and the other two sides are 8 feet.

   b. If the base has length b, and the other two sides have length s.

   c. What % shorter is the minimum distance than the 2 side distance in (a)?

3. For a scalene triangle, find the minimum distance using the Steiner Point <u>or</u> write a computer program to compute the distance <u>or</u> devise a new method to compute or measure the distance.
   a. If the sides are 4, 6, and 8.

   b. If the sides are a, b, and c.

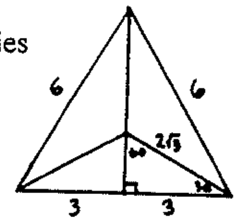
   c. What % shorter is the minimum distance than the 2 side distance in (a)?

4. Generalize the above to quadrilaterals (square, rectangle, trapezoid).

[Farber]

STEINER ACTIVITY #8 - SOLUTIONS

1. (a) Given an equilateral triangle with side 6. Use the 30, 60, 90 triangle, which has sides with ratios $1, \sqrt{3}, 2$. See diagram. Since the side opposite 60 must be 3 ($1/2s = 1/2(6) = 3$), we must multiply all sides in the ratio by $\sqrt{3}$. So the hypoteneuse is $2\sqrt{3}$. So the Steiner sum is 3 times the length of the hypoteneuse, $3(2\sqrt{3}) = 6\sqrt{3}$.

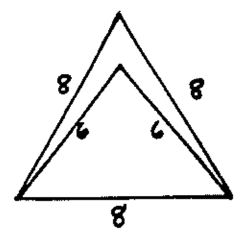


   (b) If in the above calculation, we used s in place of 6, the Steiner sum would be $s\sqrt{3}$.

   (c) $(12-6\sqrt{3})/12 = 1 - \sqrt{3}/2 = .134 = 13.4\%$.
   In the general case, it is $(2s - s\sqrt{3})/2s = 1 - \sqrt{3}/2 = 13.4\%$.

2. (a) Construct an equilateral triangle on the base and use the result of 1(b); then (the Steiner sum for an isosceles triangle) = (the Steiner sum for the equilateral triangle + altitude of isosceles triangle - altitude of the equilateral triangle) $= 8\sqrt{3} + \sqrt{20} - 4\sqrt{3} = 4\sqrt{3} + 2\sqrt{5} = 6.9282 + 4.4721 = 11.4003$.

   (b) Steiner sum is:
   $AS + BS + CS = 2c + (BM - a) =$
   $2(2a) + BM - a = 3a + BM =$
   $3(b/(2\sqrt{3})) + \sqrt{s^2 - (b/2)^2} =$
   $\sqrt{3}b/2 + \sqrt{s^2 - (b/2)^2}$



   (c) $(12 - 11.4003)/12 = 4.9975\%$.

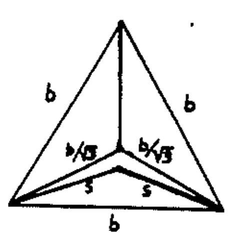
****Suggestion**** *At this time introduce an example with no Steiner Point. [b = 9 and s = 5]. Using the method of (a), it might appear that there is a solution when one does not exist indicating the need for a test to determine esistence. TEST 1: [Use sin(V/2) = (b/2)/s to find the vertex angle. If V is more than 60 there is no solution.] TEST 2: [If the altitude of the isosceles triangle is < distance from the base to the Steiner Point of the equilateral triangle there is no solution; i.e., if*

$$\sqrt{s^2 - (b/2)^2} < 1/3(b\sqrt{3}/2) = b\sqrt{3}/6,$$

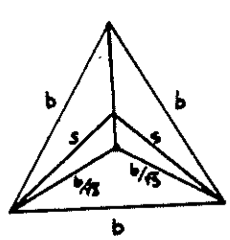
i.e., if $s < b/\sqrt{3}$.

CASE	*STEINER SUM*	*2-SIDE SUM*
1. $s < b/\sqrt{3}$	*No Steiner Point*	—
2. $s < b$ *and* $s > b/\sqrt{3}$	$b\sqrt{3} + \sqrt{s^2 - (b/2)^2} - b\sqrt{3}/2$	$2s$
3. $s > b$	*Same as case 2*	$b + s$

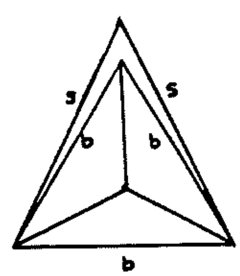
CASE 1



CASE 2



CASE 3



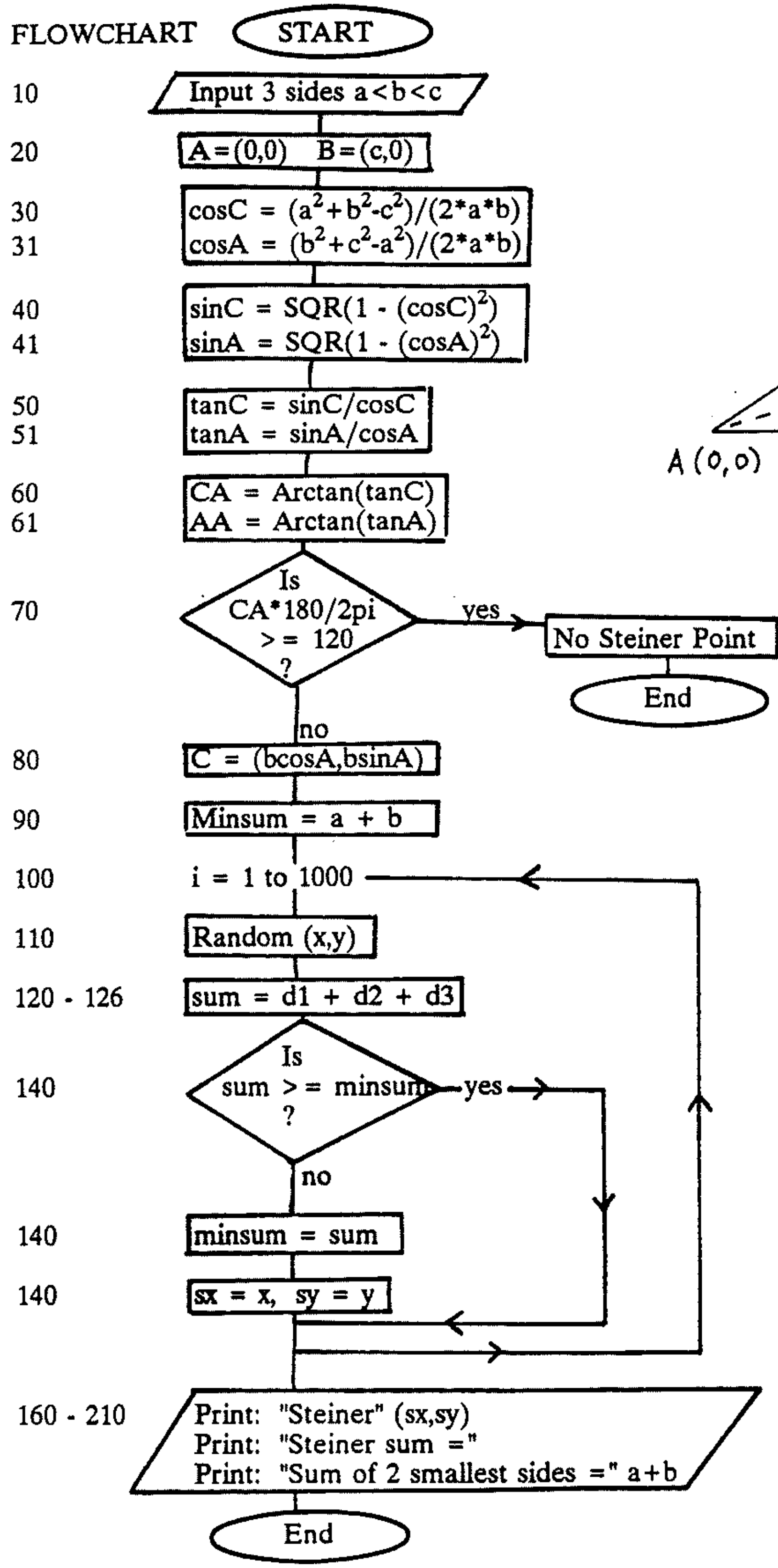
3. Scalene Triangle Suggestions:
Existence: Use the Law of Cosines to find the largest angle; if it is less than 120, a solution exists. In the 4, 6, 8 case, cosC = $(4^2 + 6^2 - 8^2)/(2*4*6)$ and A = 116.09, thus it exists. Evaluation of the Steiner sum is a difficult problem. Allow students to struggle with it until they realize it is a difficult problem that needs a method of approximation. There are several possible approximation methods.

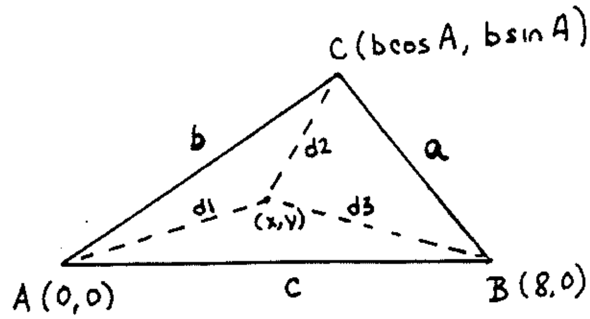
(1) Use a scale drawing of the triangle. On a plastic sheet draw 3 rays with a common endpoint and with 120 degree angles between them. Slide it around to locate the Steiner Point, by placing each ray over a vertex. Measure the distances and get the sum, etc.

(2) Set up the triangle ABC on a coordinate system A = (0,0), B = (8,0), and C = (6cosA,6sinA) = (21/4, 135/4) = (5.25,2.90473751) where angle A can be determined by the Law of Cosines and trigonometry. Randomly generate many points inside the triangle (or a rectangle containing the triangle) and generate the sum using the distance formula. Keep track of the coordinates for the smallest sum and print the result.

The following two pages contain a flowchart and a computer program which will approximate the Steiner Point by this method for any triangle with side lengths a, b, and c.

FLOWCHART


START
10 Input 3 sides a<b<c
20 A=(0,0) B=(c,0)
30 cosC = (a^2+b^2-c^2)/(2*a*b)
31 cosA = (b^2+c^2-a^2)/(2*a*b)
40 sinC = SQR(1 - (cosC)^2)
41 sinA = SQR(1 - (cosA)^2)
50 tanC = sinC/cosC
51 tanA = sinA/cosA
60 CA = Arctan(tanC)
61 AA = Arctan(tanA)
70 Is CA*180/2pi >= 120 ?
yes
No Steiner Point
End
no
80 C = (bcosA,bsinA)
90 Minsum = a + b
100 i = 1 to 1000
110 Random (x,y)
120 - 126 sum = d1 + d2 + d3
140 Is sum >= minsum ?
yes
no
140 minsum = sum
140 sx = x, sy = y
160 - 210 Print: "Steiner" (sx,sy)
Print: "Steiner sum ="
Print: "Sum of 2 smallest sides =" a+b
End


C (bcos A, bsin A)
b
d2
a
d1
(x,y)
d3
A (0,0)
c
B (8,0)

```
5  REM   Steiner Point and sum approximation for (scalene) triangle
10 INPUT "Type in 3 sides from smallest to largest ";A,B,C
20 A1=0 : A2=0 : B1 = C : B2 =0
25 REM  statements 30 - 61 compute angles A and C in radians
30 COSA = (B^2+C^2-A^2)/(2*B*C)
31 COSC = (A^2+B^2-C^2)/(2*A*B)
40 SINA = SQR(1-(COSA)^2)
41 SINC = SQR(1-(COSC)^2)
50 TANA = SINA/COSA
51 TANC = SINC/COSC
60 AA = ATN(TANA)
61 CA = ATN(TANC)
65 IF COSC < 0 THEN CA = CA + 3.1415926#
67 GOTO 80
70 IF CA*180/(3.1415926#) >= 120 THEN PRINT "No Steiner Point': END
80 CP1 = B*COS(AA) : CP2 = B*SIN(AA)
85 REM Initialize (SX,SY) to be the vertex between smallest two sides.
90 MINSUM = A+B
92 SX = CP1
93 SY = CP2
95 REM  The loop tries 1000 random points in the rectangle about the triangle.
100 FOR I = 1 TO 1000
110    RANDOMIZE TIMER : X = C*RND : RANDOMIZE TIMER : Y = C*RND
120    D1 = SQR(X^2+Y^2)
121    D2 = SQR((X-C)^2+Y^2)
122    D3 = SQR((X-CP1)^2+(Y-CP2)^2)
124    SUM = D1 + D2 + D3
140    IF SUM < MINSUM THEN MINSUM = SUM :  SX = X :  SY = Y
150 NEXT I
160 PRINT "By random generation of points to get approximations"
165 PRINT "given sides of length "A;B;C
167 PRINT "Vertices are A=(0,0), B=("C",0), C=("CP1","CP2")"
170 PRINT "Steiner Point = ("SX","SY")"
180 PRINT "Steiner sum = " MINSUM
190 PRINT "sum of the 2 smallest sides = " A+B
200 PRINT "The sum of 2 smallest sides - the Steiner sum = " A+B-MINSUM
210 PRINT "The % improved by Steiner point = " (A+B-MINSUM)/(A+B)*100"%".
300 END
```

4. Generalizations

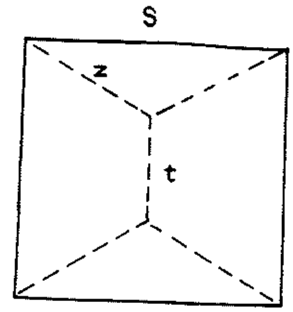
SQUARE of side s:

Sum = 4z + t (s/2)/z = $s/\sqrt{3}$.

Thus, z = $s/\sqrt{3}$. But t = s - 2(z/2) = s-z.

Hence, sum = 4z + (s-z) = 3z + s = $\sqrt{3}s$ + s.

(There are 2 sets of 2 Steiner Points).



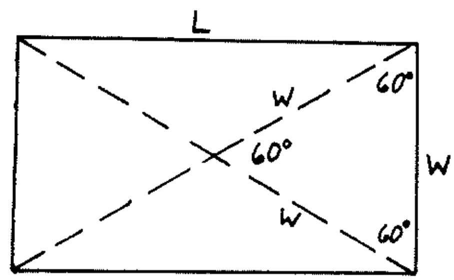
RECTANGLE, Length (L) > Width (W):

CASE 1: L = $\sqrt{3}$W There is only one Steiner Point. It is at the center. Steiner sum = sum of the diagonals = 4W.
Sum of 3 shortest sides = 2W + $\sqrt{3}$W < 4W.

CASE 2: L < $\sqrt{3}$W There are 2 Steiner Points whose segment is parallel to the width.
Steiner sum = 4z + t = 4z + W - z = 3z + W = $3(L/\sqrt{3})$ + W = $\sqrt{3}$L + W.
Sum of 3 shortest sides: L + 2W.

CASE 3: L > $\sqrt{3}$W There are 2 Steiner Points whose segment is parallel to the length.
Steiner sum = 4z + t = 4z + L-z = 3z + L = $3(W/\sqrt{3})$ + L = $\sqrt{3}$W + L
Sum of 3 shortest sides = L + 2W

Note: Case 2 and 3 have diagrams similar to that of the square above.



CASE 1

TEACHING NOTES
Steiner Points - Calculus

Prerequisite Skills: The skills necessary are inductive and deductive reasoning, algebra, geometry, trigonometry, analytic geometry, and differential calculus skills.

Objective: The student will be able to:

1. *Understand the concept of minimum network distance and what Steiner Points are.*

2. *Apply maximum/minimum techniques to find minimum network distance and Steiner Points.*

3. *Generalize results from number 2 to hypothesize the location of Steiner Points for a rectangle.*

4. *Generalize results to hypothesize the particular set of Steiner Points for a rectangle that will yield the minimum network distance.*

5. *Generalize results to hypothesize the conditions in which a rectangle will yield two, one, or no Steiner Points.*

6. *Test the hypotheses from numbers 3 and 5 using soap bubbles.*

Materials

- *Mathematical models for constructing minimum network distances (a square, rectangles with 4:3, $\sqrt{3}$:1, and 2:1 ratios are necessary, other quadrilaterals, pentagons, and hexagons would be beneficial).*
- *Soap and glycerine.*
- *Water.*
- *Straws.*
- *Compasses, straightedges, and protractors.*
- *Calculators.*

Discussion

Using maximum/minimum techniques, students will see that a square will yield two sets of Steiner Points (one "horizontal" and one "vertical"), both of which will lead to the minimum network distance. As one side is made a little longer, the two sets of Steiner Points will remain but the points parallel to the longer side will yield the shorter minimum network distance. However, for a while, the other pair will result in a shorter network distance than if one used the "outside" network distance. This phenomenon can lead to a classroom discussion of what "local" or relative maximum/minimum means. Both sets of Steiner Points lead to relative minima but only one is the absolute minimum for a non-square rectangle. (When students conduct the soap bubble experiments, they should also notice that the soap "settles" into certain paths through certain points because these paths are relative minima.)

[Zangari, Reynolds]

As one side of the rectangle is made increasingly longer, the Steiner Points parallel to the longer sides remain fixed relative to the shorter sides. The network distance increase is only from the increase in distance between the two Steiner Points. The Steiner Points parallel to the shorter sides will move closer together until at a certain ratio between longer and shorter sides ($\sqrt{3}$ to 1), there is just one Steiner Point. When this happens, even though a relative minimum for network distance occurs there, this distance is larger than the "outside" network distance.

When the ratio between longer and shorter sides becomes greater than $\sqrt{3}$ to 1, there is only one set of Steiner Points and this set is parallel to the longer sides (and it yields the minimum network distance).

Besides what is covered in the previous paragraphs, it is hoped that students will see that a Steiner Point is always connected by three line segments, all at 120 degrees to each other. Once this is discovered, then it is clear why the set of Steiner Points "disappears" when the ratio between sides reaches a certain value.

Students will often have difficulty recognizing the 120 degree relationship until they conduct experiments with soap bubbles. Once they realize that the three line segments meeting at a Steiner Point do so at 120 degrees, they are able to predict, somewhat, the location of Steiner Points for five or six points. With some help, they should be able to determine that the total number of possible Steiner Points is $n - 2$, where n is the number of points to be connected.

Suppose a rectangle is divided into four triangles by two diagonals. One could find the Steiner Point for each triangle and it would be the same Steiner Point for the rectangle. This suggests a method for finding Steiner Points for any polygon.

STEINER ACTIVITY #9
Calculus

For the following seven problems, students are to

1. Set up a function t find the minimum network distance.
2. Determine the minimum network distance and compare with the shortest "outside" network distance.
3. Locate the Steiner Point in terms of x.

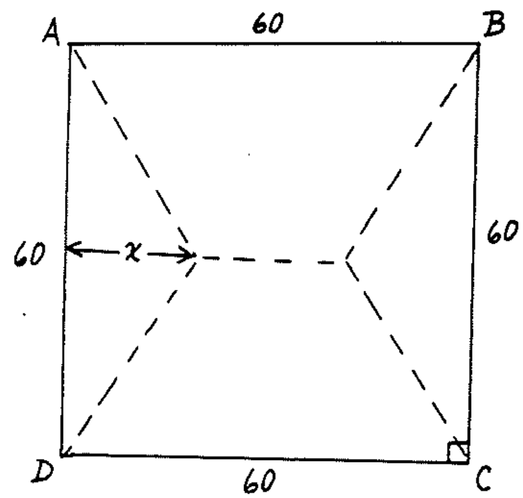
After completing all seven problems, students are to generalize the following for a rectangle.

1. Predict which pair of Steiner Points would result in the minimum network distance.
2. Predict when only one set of Steiner Points will occur.
3. Predict when there would be only one Steiner Point.
4. Identify the feature of the Steiner Point that remains consistent.

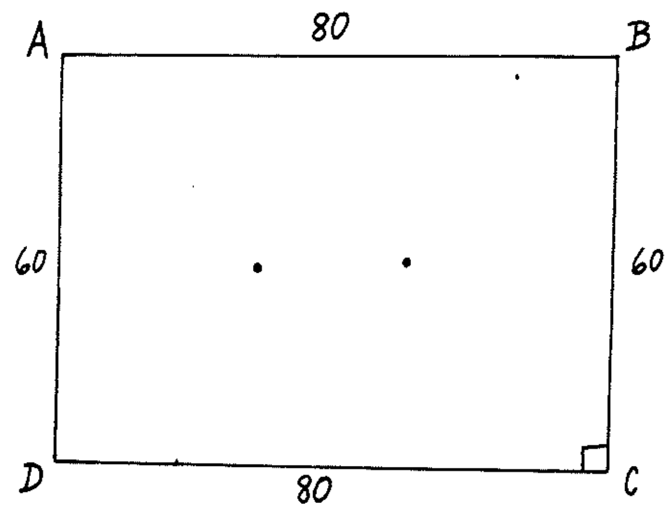
CALCULUS: Max/Min - Steiner Points

Use max/min techniques to find the shortest distance (path) connecting pointss A, B, C, and D using Steiner Points. In each example, compare this distance with the minimum path on the "outside". (Express answers in radical form, if appropriate, and then convert to three place decimal form.) Throughout the examples, there is a pattern for determining x. If possible, discuss this and show how that causes problems with the Steiner Points for the last problem.

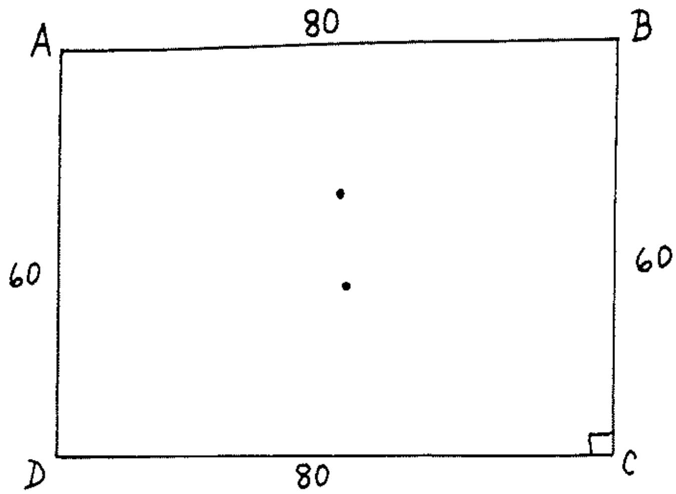
1.



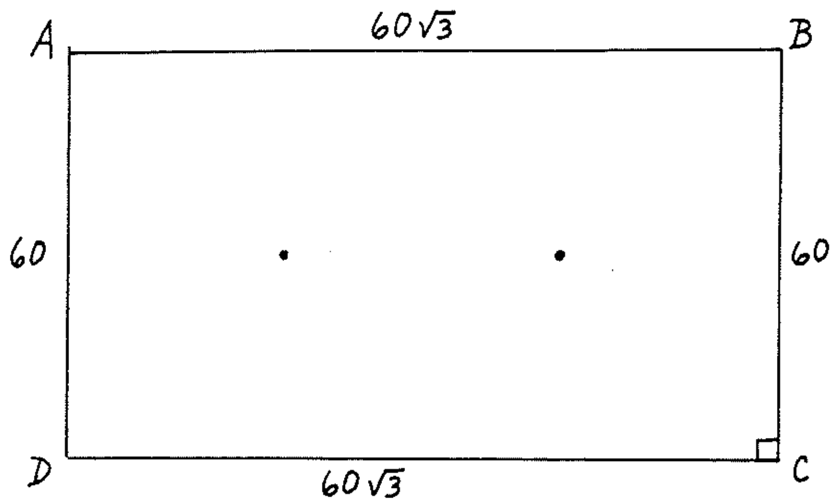
2.



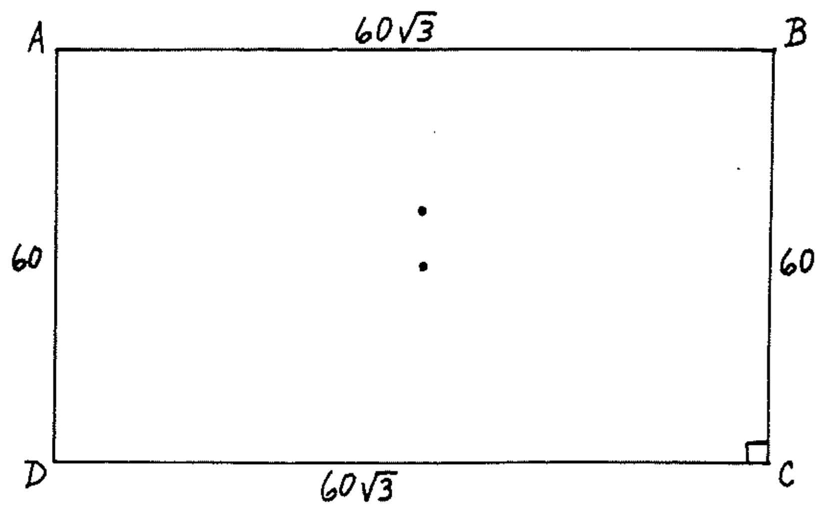
3.


A
80
B
60
60
D
80
C

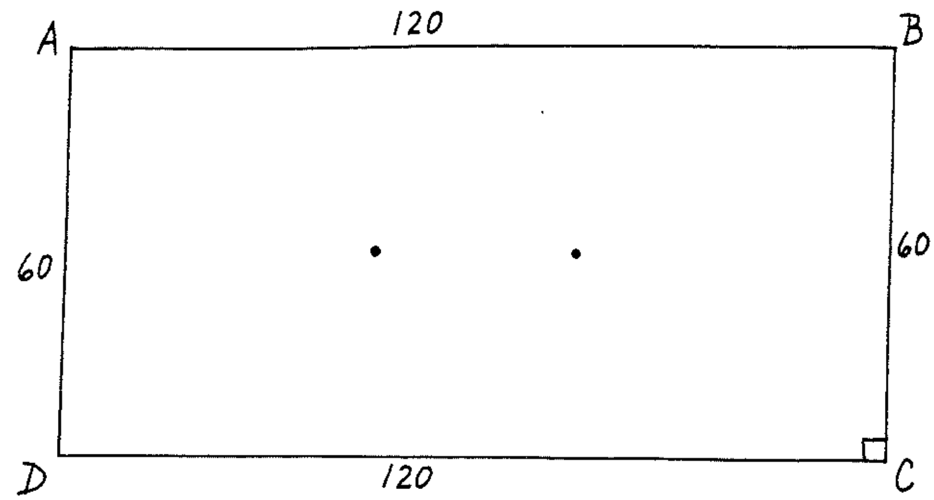
4.


A
$60\sqrt{3}$
B
60
60
D
$60\sqrt{3}$
C

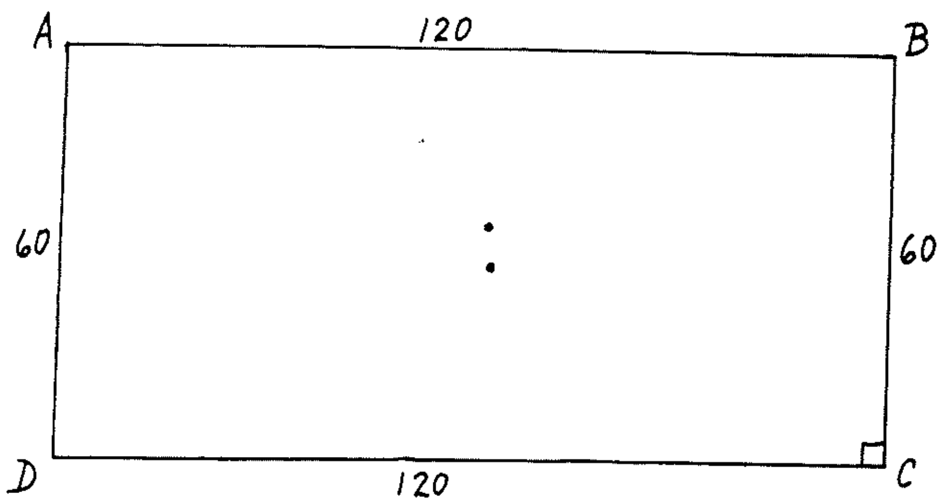
5.


A
$60\sqrt{3}$
B
60
60
D
$60\sqrt{3}$
C

6.



7.



Answers: Steiner Activity #9

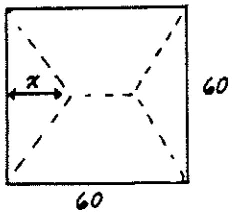
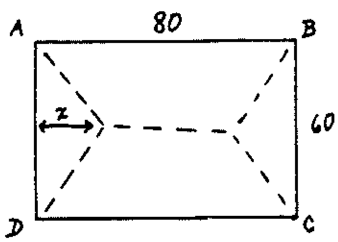
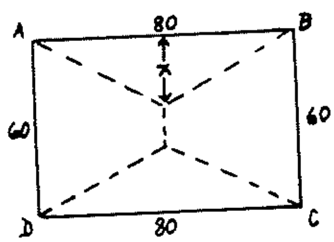
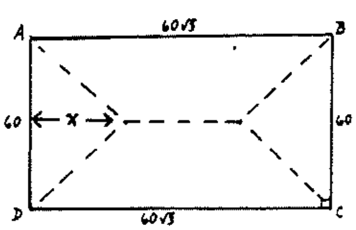
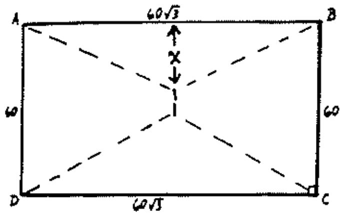
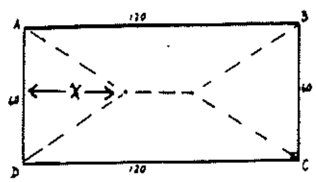
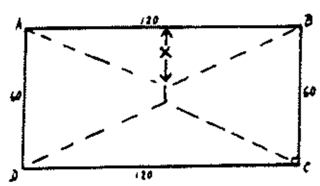
Figure and Ratio	Distance	Range of x	Exterior Distance
1.  Ratio 1 : 1	$D = 4\sqrt{900+x^2}+60-2x$ $D' = \frac{4x}{\sqrt{900+x^2}} - 2 = 0$ $x = 10\sqrt{3}$ $D = 60\sqrt{3} + 60 \approx 163.9$	$0 \le x \le 30$	180
2.  Ratio 3 : 4	$D = 4\sqrt{900+x^2}+80-2x$ $D' = \frac{4x}{\sqrt{900+x^2}} - 2 = 0$ $x = 10\sqrt{3}$ $D = 60\sqrt{3} + 80 \approx 183.9$	$0 \le x \le 40$	200
3.  Ratio 3 : 4	$D = 4\sqrt{1600+x^2}+60-2x$ $D' = \frac{4x}{\sqrt{1600+x^2}} - 2 = 0$ $x = \frac{40\sqrt{3}}{3}$ $D = 80\sqrt{3} + 60 \approx 198.6$	$0 \le x \le 30$	200
4.  Ratio $1 : \sqrt{3}$	$D = 4\sqrt{900+x^2}+60\sqrt{3}-2x$ $D' = \frac{4x}{\sqrt{900+x^2}} - 2 = 0$ $x = 10\sqrt{3}$ $D = 120\sqrt{3} \approx 207.8$	$0 \le x \le 30\sqrt{3}$	223.9

Figure and Ratio	Distance	Range of x	Exterior Distance
5.  Ratio $1 : \sqrt{3}$	$D = 4\sqrt{2700+x^2}+60-2x$ $D' = \frac{4x}{\sqrt{2700+x^2}} - 2 = 0$ $x = 30$ $D = 4\sqrt{2700+900} = 240$	$0 \leq x \leq 30$	223.9
6.  Ratio 1 : 2	$D = 4\sqrt{900+x^2}+120-2x$ $D' = \frac{4x}{\sqrt{900+x^2}} - 2 = 0$ $x = 10\sqrt{3}$ $D = 60\sqrt{3} + 120 \approx 223.9$	$0 \leq x \leq 60$	240
7.  Ratio 1 : 2	$D = 4\sqrt{3600+x^2}+60-2x$ $D' = \frac{4x}{\sqrt{3600+x^2}} - 2 = 0$ $x = 20\sqrt{3}$ Not in domain! If s=30 is used (endpoint), then $D=4\sqrt{4500}=120\sqrt{5}$ ≈ 268.3 which is greater than 240.	$0 \leq x \leq 30$	240

STEINER ACTIVITY #10
Calculus

The objective of this activity is to test the predictions made in Activity 9 for locating the minimum distance network of four points making a rectangle and for determining when and how many pairs of Steiner Points will exist for a rectangle. Models other than rectangular can then be used to test conjectures made from the rectangular examples.

Exercises

1. Start with the square model and dip it in the soapy water, keeping one side of the square down while lifting the model from the soapy water. (A film of soap should form lines connecting the three sides not kept down.) Using a straw, blow gently at one side that has the film of soap. (The soap should move to an equilibrium stage which should locate the two Steiner Points.) Either redip the model or blow the present film back to the four rectangular points. Then, blow on another side such that the Steiner Points located are perpendicular to where the first pair were. Do the locations seem to match your calculations from Activity 9? Explain.____________________

2. Dip the non-square rectangular models and follow the same procedure as in number 1. Start with the 4:3 ratio rectangle first, followed by the $\sqrt{3}$:1 and 2:1 ratio ones. Explain fully what you find with respect to number of Steiner Points and number of pairs of Steiner Points. How do these observations compare with your predictions in Activity 9?_______

3. At a Steiner Point, what are the measures of the angles that are formed there?________

4. How does the observation in number 3 relate to the number of Steiner Points for a $\sqrt{3}$:1 or a 2:1 ratio rectangle? Explain.____________________

5. Sketch a quadrilateral that will have no Steiner Points.

6. Dip other models - quadrilaterals, pentagons, hexagons - and try to find all the Steiner Points. For any pentagon, what is the maximum number of Steiner Points that can occur at one time?__________ For any hexagon?____________ Generalize these results for a polygon on n sides.____________________

Chapter 3: COLORING APPLICATIONS

COLORING APPLICATIONS
TABLE OF CONTENTS

One major source of the applications of graph theory arises from graph colorings. Several teachers contributed material on this subject, which has a natural attraction for students and teachers alike. Most of the activities in this section are accessible to students in all levels of mathematics.

References: *Copes, Wayne, Clifford Sloyer, Robert Stark, and William Sacco; Graph Theory: Euler's Rich Legacy; Janson Publications, Providence, RI, 1987.*

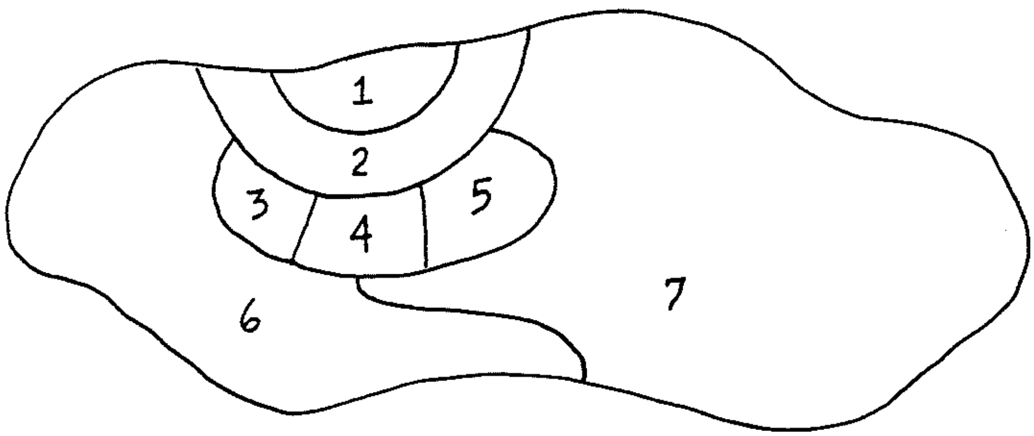
Fisher, John (editor); The Magic of Lewis Carroll; Simon and Schuster, New York, 1973.

Francis, Richard; Module 13: The Mathematician's Coloring Book; COMAP, Mass., 1989.

TEACHING NOTES:
I. Motivation and Introductory Materials

LOOK KIDS -- Mathematicians Color!

The question of how many different colors are needed to color a map so that no 2 countries with a common border have the same color is an old one. Of course, if we have 20 countries, we could use 20 different colors, but cartographers , being misers, wanted to know the fewest number of colors they needed for any map. Try this one:



Try red for region 1. We can not use red for region 2, so use blue. Regions 3, 4, and 5 can not be blue, but one or two of them can be red. Color region 4 red and use green for regions 3 and 5, since they can be any color but red or blue. Regions 6 and 7 can not be colored red, green, or blue, and they must be different colors. Color region 6 yellow and change region 5 to yellow. Color region 7 green. We have colored this "map" in 4 colors.

After many years , mathematicians, with the help of computers, finally proved that 4 is the minimum number of colors needed for any such map.

So why are mathematicians still coloring today? They are applying coloring to many problems involving scheduling, such as committee meetings, college courses, and product manufacture.

[Van Hise]

TEACHING NOTES:

II. History of the 4-color conjecture.

A. Proposed by Moebius in 1840, Frederick Guthrie, a student of DeMorgan 1850, and Cayley 1878.
B. "Proof" by Kempe, a British Lawyer, in 1879, proved faulty by Heawood in 1890. P.J. Heawood showed that 5 colors are always sufficient.
C. Philip Franklin (1920): 25 regions can be done with 4 colors.
D. C.N. Reynolds (1926): 27 regions can be done with 4 colors.
E. Franklin (1936): 31 regions can be done with 4 colors.
F. C.E. Winn (1943): 35 regions can be done with 4 colors.
G. Ore & Stemple (1968): 40 regions can be done with 4 colors.
H. Kenneth Appel & Wolfgang Haken computer proof 1976:
- *several hundred pages.*
- *1000 hours of computer calculation.*
- *1936 reducible configurations, each with a search of up to a half million logical options to verify reducibility.*
- *nearly a billion calculations.*

Do you think a proof which requires a computer is really a proof? Justify your position.

Thought: Are there other math problems that require a computer and cannot be solved by humans alone?

Applications:

A. Time sharing in computer science.
B. Phasing traffic lights.
C. Scheduling & maintenance problems in operations research.
D. Design of printed circuits in Electrical Engineering.

[Devino]

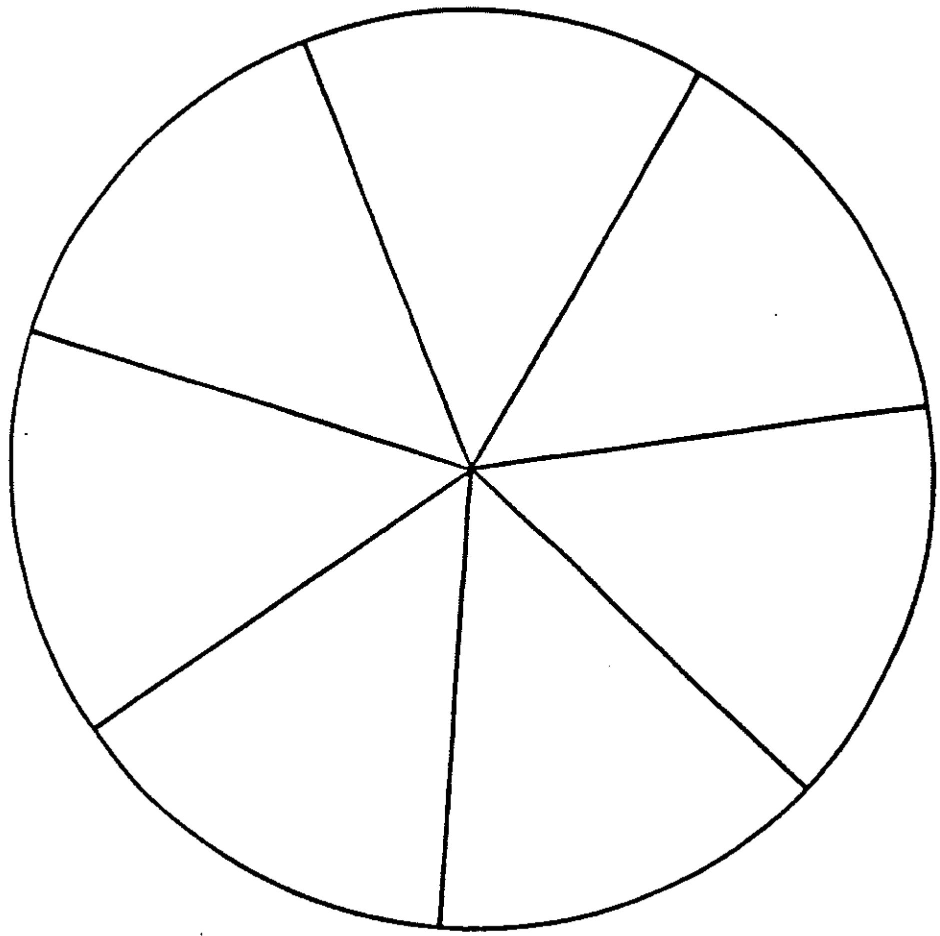
III. Some graphs take fewer than four colors, some take more. The activities on the following pages are for student discovery. Give an activity to the students as homework, prior to any coloring talk. Instruct the students to:

1. *Color each region so that no two of the same color share a border.*
2. *Use the least number of colors.*
3. *Answer all questions.*

[Coulter, Simon]

COLORING ACTIVITY #1
Two-Dimensional regions

A Colorful Pie



Color the pie graph above so that no two pieces of the same color touch along a border.

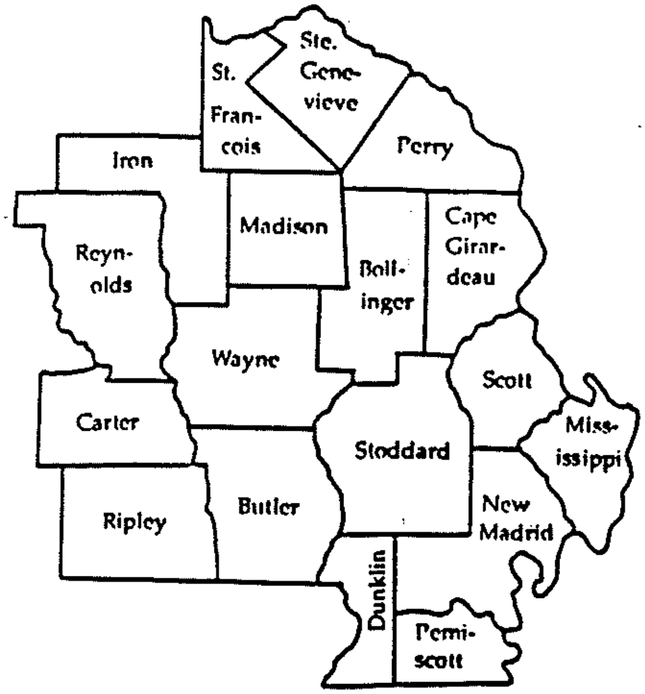
1. Did you need as many as seven different colors? Note that pieces of the same color may touch at a single point.

2. Could the coloring have been done with only three different colors?

3. Find a formula for coloring a pie with n pieces (above, n=7). Can one formula work for n both even and odd?

[FRANCIS]

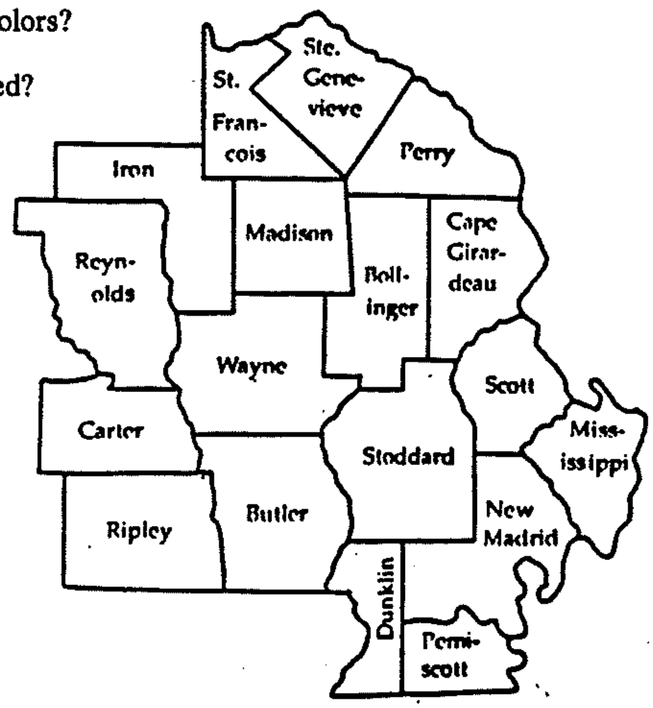
Colorful Southeast Missouri



Color the eighteen counties above so that no two having the same color touch along a border.

1. Did you need as many as five different colors?

2. What is the least number of colors needed?



3. What is the least number of counties through which one must pass in driving from Cape Girardeau County to Dunklin County? Color this set of counties with the least number of colors possible.

[FRANCIS]

A Colorful Chain of States (Odd)

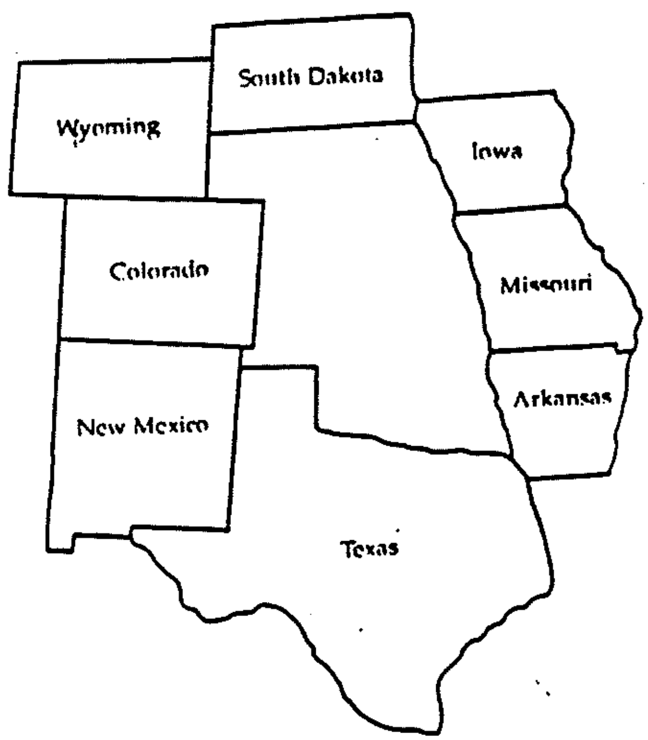
Color the states to the left so that no two of the same color touch along a border.

Can this closed chain of an ODD NUMBER of states be colored with but two colors, or will at least three colors be needed?

A Colorful Chain of States (Even)

Color the states to the right so that no two of the same color touch along a border.

Can this closed chain of an EVEN NUMBER of states be colored with but two colors, or will at least three colors be needed?



[FRANCIS]

TEACHING NOTES:
IV. Group Work - discoveries about 3-dimensions.

Set up small groups at work stations with coloring markers or frosting, and assorted objects to color, such as:

Bagels
Mobius strip
Cylinder (can)
Tetrahedron
Hexahedron
Octahedron
Sphere (an orange will work)

Let Coloring Activity #2 be used to guide discovering and finish concluding questions for homework. Students should be encouraged to design coloring patterns of their own.

[Coulter, Simon]

Editor's Note:
Let students cover a paper with intersecting circles. Then discover how many colors are necessary to color the regions formed. (The answer is a surprise -- why?,

COLORING ACTIVITY #2
3-Dimensions

Worksheet: Coloring Experiment

What shape was used:

Names of experimenters:

1. Before starting the experiment, answer the following questions:

   A. How many different colors do you believe will be needed to color the regions so that:
      a) you use as few colors as possible and
      b) no two regions of the same color will touch along a border (the same color regions may touch at one point)?

   B. What colors are you going to use and how do you plan to set up this experiment?

2. Answer the following questions after you have finished the experiment:

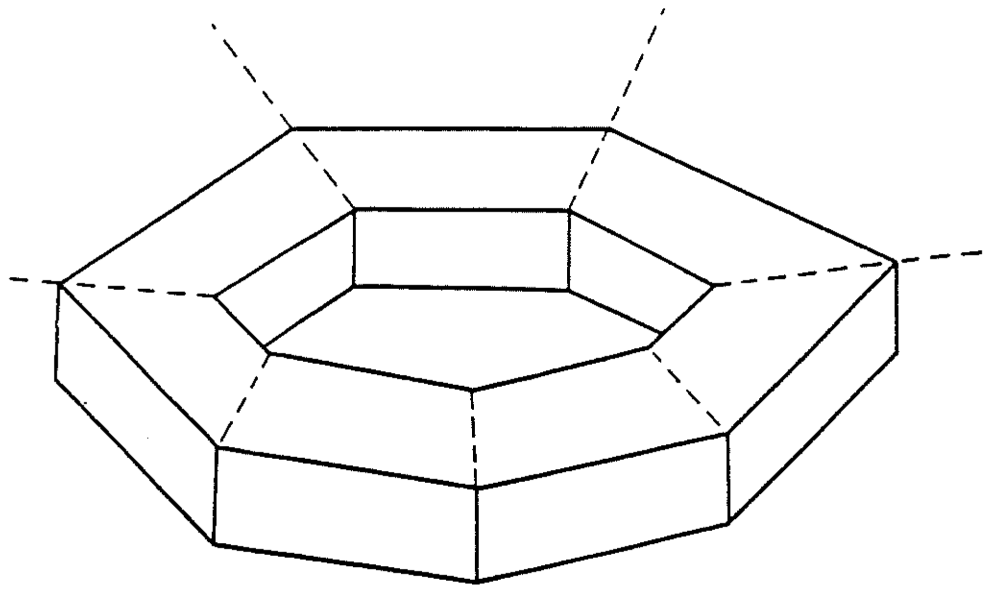
   A. How many colors were needed?

   B. Did you use more or less colors than you thought from question number 1? Why?

3. Write down any conclusions you have about three dimensional objects and colors verses the previous coloring assignment on two dimensional surfaces.

[Coulter, Simon]

Doughnut



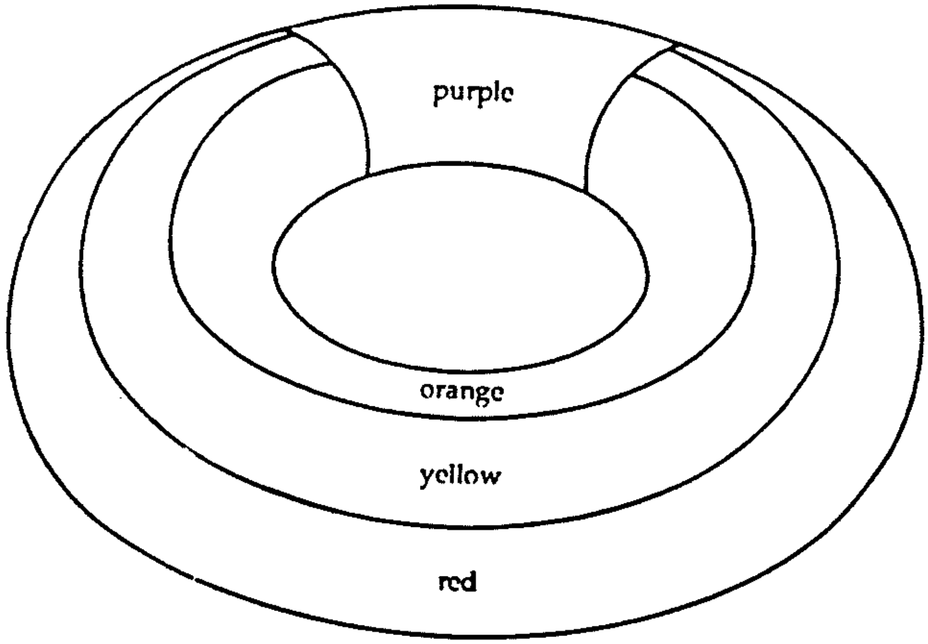
Divide the doughnut into the regions shown.

1. How many colors did you use to color the doughnut so no regions with the same color border each other?

2. Did you use the minimum number of colors possible? If not, what do you think would be the minimum number?

[FRANCIS]

Colorful Jelly Doughnuts



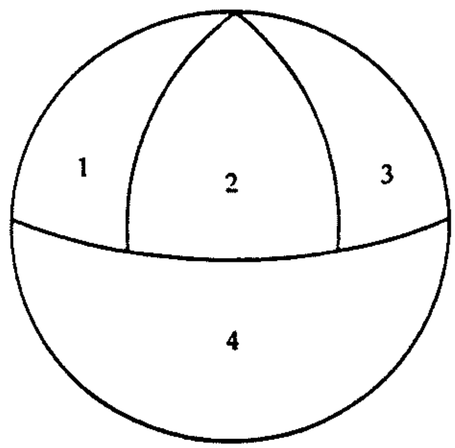
Consider the strawberry jelly (RED), the peach jelly (YELLOW), the orange jelly (ORANGE), and the grape jelly (PURPLE), which cover the above doughnut.

1. Do all of the jelly types touch all of the other jelly types along a border?

2. Explain how the red strawberry jelly touches the orange jelly along a border.

Colorful Golf Ball

A manufacturing company wishes to produce a golf ball so that four different colors cover the ball and moreover, each solid color touches all the other colors along some border.

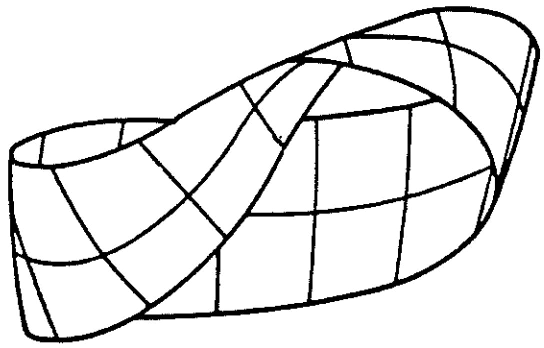


Use an orange and markers or tubes of frosting in your experiment. An orange is easier to handle than a golf ball.

1. Make regions on the golf ball as shown in the figure above. Continue the regions on the back side of the golf ball.

2. Suppose the manufacturer wanted to do the same with five unbroken colors. Can you show him how to do it? Or must you say it is impossible?

Mobius Band

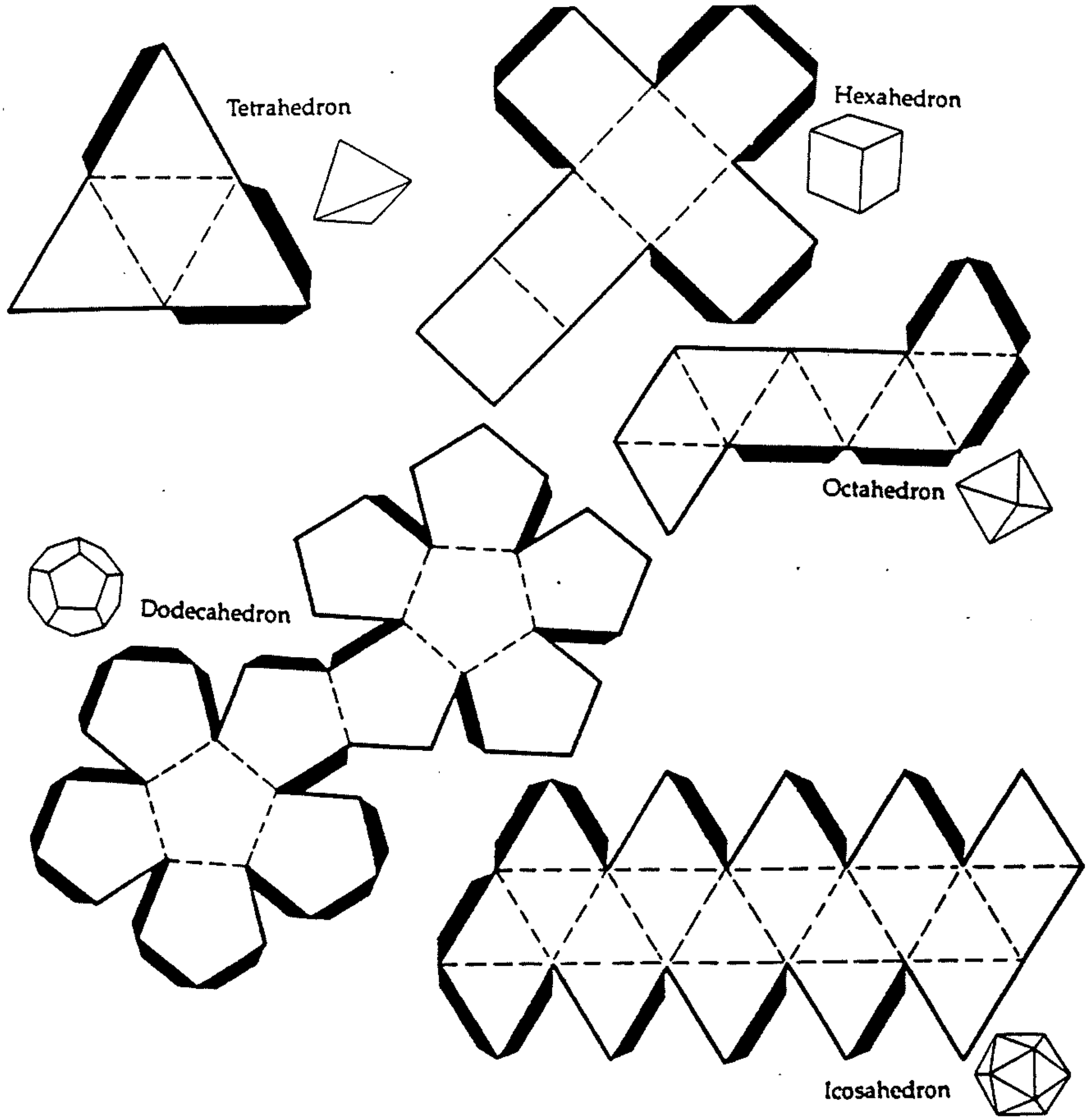


Construct a Mobius band as shown. Color the regions so that no regions with the same color border each other.

1. How many colors did you use?

2. Was it possible to use four colors?

1. Construct the five regular polyhedra shown here.



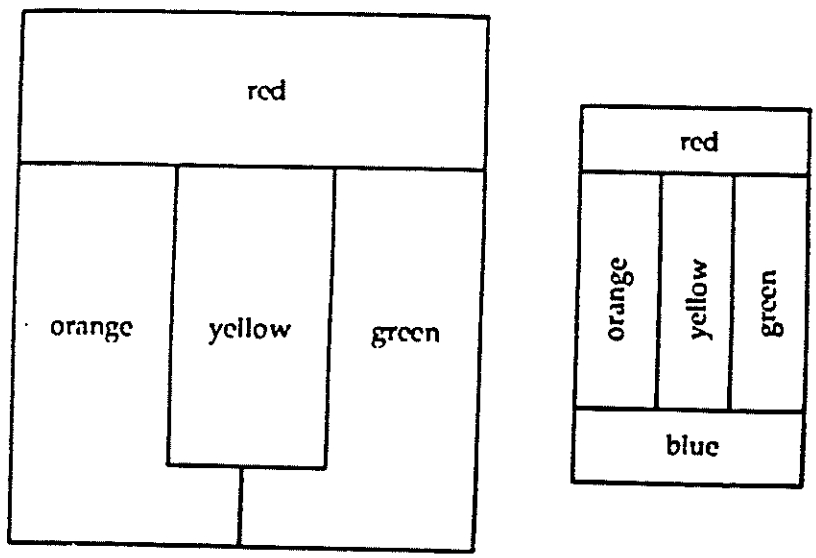
2. The four-faced polyhedron is called a tetrahedron. Draw four regions (countries) which cover the polyhedral surface completely so that:

    a. Only three colors are needed in shading.

    b. Four colors are needed in shading.

COLORING ACTIVITY #3
Mixed Practice

Colorful Carpeting 1



Mr. Rugman wishes to display his four strips of differently colored carpet samples on a floor. Moreover, he wants each solid piece to touch all the others along a border, so as to make color comparisons easy for his customers as well as to have an eye-catching display.

1. Color the four strips of carpeting above so that Mr. Rugman will be pleased with the color matching.

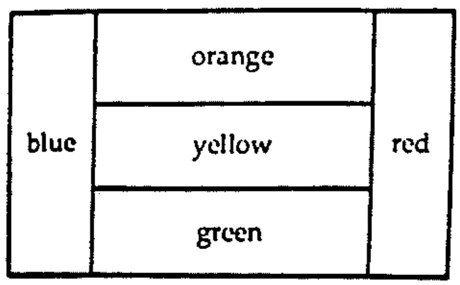
2. Later, Mr. Rugman wishes to do the same for five differently colored strips (each to be kept in one solid piece). What is wrong with the arrangement on the right?

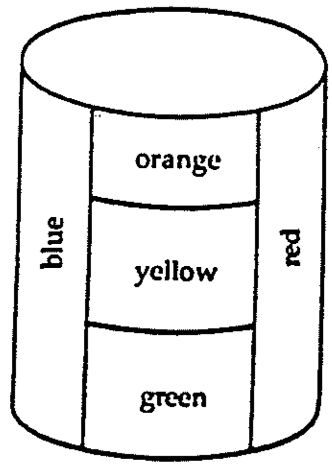
3. Can you show Mr. Rugman how to arrange his five carpet pieces on the floor? Or must you tell him that it can't be done?

[FRANCIS]

Colorful Carpeting Rolls 2

Suppose Mr. Rugman decides the five carpet samples need not lie flat on the floor as shown here, but rather made into a curved surface (cylinder), so that the red now touches the blue along a border. Consider the figure below.





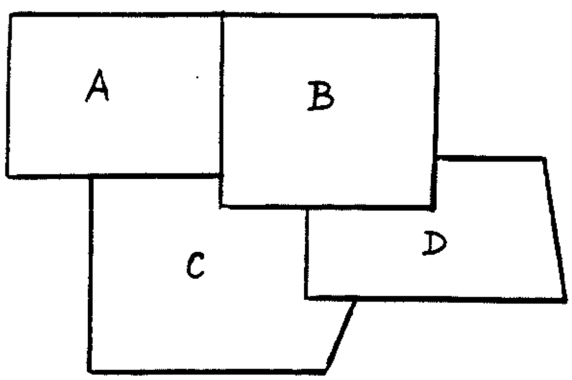
1. The red now touches the blue on the back side of the roll. What two colors still don't touch?
2. Suppose the orange and green are now extended on the underside of the rug arrangement. Will each of the five colors now touch all the others along a border? Do the examples shown here, suggest that what is IMPOSSIBLE on a flat surface may be POSSIBLE after all on another surface?
3. What would happen if the ends of the cans (rugs) could be colored?

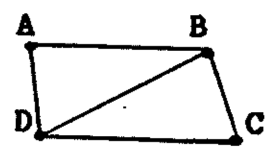
[FRANCIS]

TEACHING NOTES:
V. Chromatic numbers, planar and non-planar graphs.

The Lewis Carroll game (see the following pages) can provide focus for discussion about the coloring problem. Flat surfaces and spheres need no more than four colors. Other surfaces may need as many as seven colors. Many surfaces require less than seven but it has been proved that more than seven is never needed. The Mobius strip has a chromatic number of six.

To turn a map into a network: The vertices are the names of the regions, and the edges of the graph represent the borders. If the vertices are connected by an edge, the colors of the vertices must be different. The fewest number of colors needed to color the vertices of a graph is called its *<u>chromatic number</u>.* *For example:*



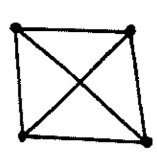


A and C may be the same color, the other two must be different colors. For example, A and C are red, B is blue, and D is green.

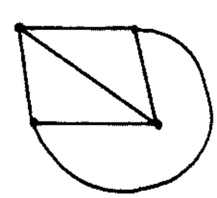
This graph, then, has three as its chromatic number.

A graph is termed *<u>planar</u>* *if it can be drawn on the plane without crossing edges. For example, these two graphs appear to be non-planar, but the left one can be redrawn. (All maps in the plane result in planar graphs).*

1.

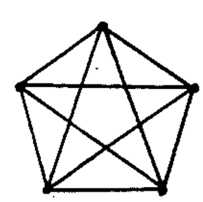


redraw as



planar
chromatic number: 4

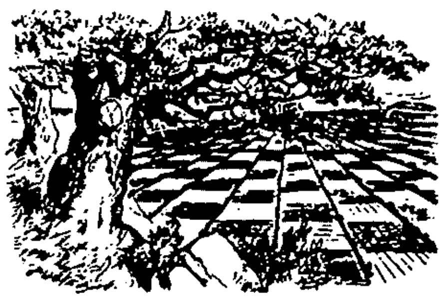
2.



non-planar
chromatic number: 5
[Coulter, Simon, Van Hise]

*The four-colour map problem**

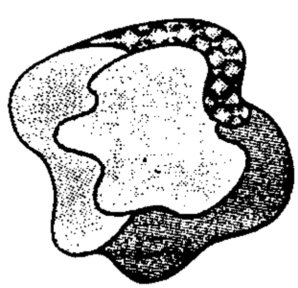
> *For some minutes Alice stood without speaking, looking out in all directions over the country - and a most curious country it was. There were a number of tiny little brooks running straight across it from side to side, and the ground between was divided up into squares by a number of little green hedges, that reached from brook to brook.*
>
> *'I declare it's marked out just like a large chess-board!' Alice said at last.*
>
> Through the Looking Glass



Another favourite puzzle of Carroll, undated by Collingwood, involved the famous topological problem of designating regions on a map by colours. Carroll saw it as a game for two players:

> *A is to draw a ficticious map divided into counties.*
> *B is to colour it (or rather mark the counties with* names *of colours) using as few colours as possible.*
> *Two adjacent counties must have* different *colours.*
> *A's object is to force B to use as* many *different colours as possible. How many can he force B to use?*

By 'adjacent' it is understood that Carroll meant touching along a line, as distinct from touching at a single point. With this accepted, A can easily force B to use as many as four colours:



No one has, in fact, ever found either a plane or a spherical map for which four colours are not sufficient, but it would be wrong to mistake this for a general proof that four are sufficient. Perhaps one day a map requiring five colours will be discovered. H. S. M. Coxeter, writing in Mathematics Teacher *for April, 1959, allows himself 'to make a conjecture that a map requiring five colours may be possible, but that the simplest such map has so many faces (maybe hundreds or thousands) that nobody, confronted with it, would have the patience to make all the necessary tests that would be required to exclude the possibility of colouring it with four colours.' Moebius himself made the assertion in 1840*

* *This material is an exerpt from Fisher's book.*

that it is impossible to draw a map of five countries on a plane so that each pair shares a common border (a problem equivalent to joining each of five dots placed on a plane to the others by non-intersecting straight lines), but this is a long way from proving the need for five colours on a map of 'n' countries. As the problem stands, one can assume that five colours are sufficient, four necessary. The world awaits the breakthrough in mathematics leading to the proof that four are both necessary and sufficient.

Of interest to Carroll would have been a development of his own game by science fiction writer, Stephen Barr, described by Martin Gardner, in which A first draws a country; B then colours it and adds a new country; A colours the new one and adds a third, and so on, each player colouring the area just drawn by his opponent until one is forced to lose the game...by using a fifth colour.

In both The Hunting of the Snark and Sylvie and Bruno Concluded Carroll continued to play cartographical games. In the former the Bellman brought a map of the sea which the crew could all understand:

'Other maps are such shapes, with their islands and capes!
But we've got our brave Captain to thank'
(So the crew would protest) 'that he's bought us the best -
A perfect and absolute blank!'

By way of contrast to the blank map Carrol supplied, the map of the German Professor in the later work is not lacking in one single detail:

Mein Herr looked so thoroughly bewildered that I thought it best to change the subject. 'What a useful thing a pocket-map is!' I remarked.

'That's another thing we've learned from your Nation,' said Mein Herr, 'map-making. But we've carried it much further than you. What do you consider the largest map that would be really useful?'

'About six inches to the mile.'

'Only six inches!' exclaimed Mein Herr. 'We very soon got to six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all! We actually made a map of the country, on the scale of a mile to the mile!'

'Have you used it much?' I enquired.

'It has never been spread out, yet,' said Mein Herr: 'the farmers objected: they said it would cover the whole country, and shut out the sunlight! So we now use the country itself, as its own map, and I assure you it does nearly as well.'

Jorge Luis Borges, in his essay 'Partial Magic in the Quixote', included in his Labyrinths, quotes Josiah Royce in the first volume of his work The World and the Individual (1899): 'Let us imagine that a portion of the soil of England has been levelled off perfectly and that on it a cartographer traces a map of England. The job is perfect; there is no detail of the soil of England, no matter how minute, that is not registered on the map; everything has there its correspondence. This map, in such a case, should contain a map of the map of the map, and so on to infinity.'

The artist, Mark Boyle, has more recently taken Carroll's initially wild assumption one step nearer to reality in a project entitled 'Journey to the Surface of the Earth'. Boyle's self-imposed task, described at length by Philip Oakes in The Sunday Times for September 27, 1970, involves the exact reconstruction of specific sites on the earth's surface, ranging from six to eight feet square, chosen at random by blindfolded people aiming darts at a map of the world. The query of British logician R. B. Braithwaite, in The Mathematical Gazette for July, 1932 - 'Can a symbol symbolize itself?' - was nearer to an answer than he knew.

[FISHER]

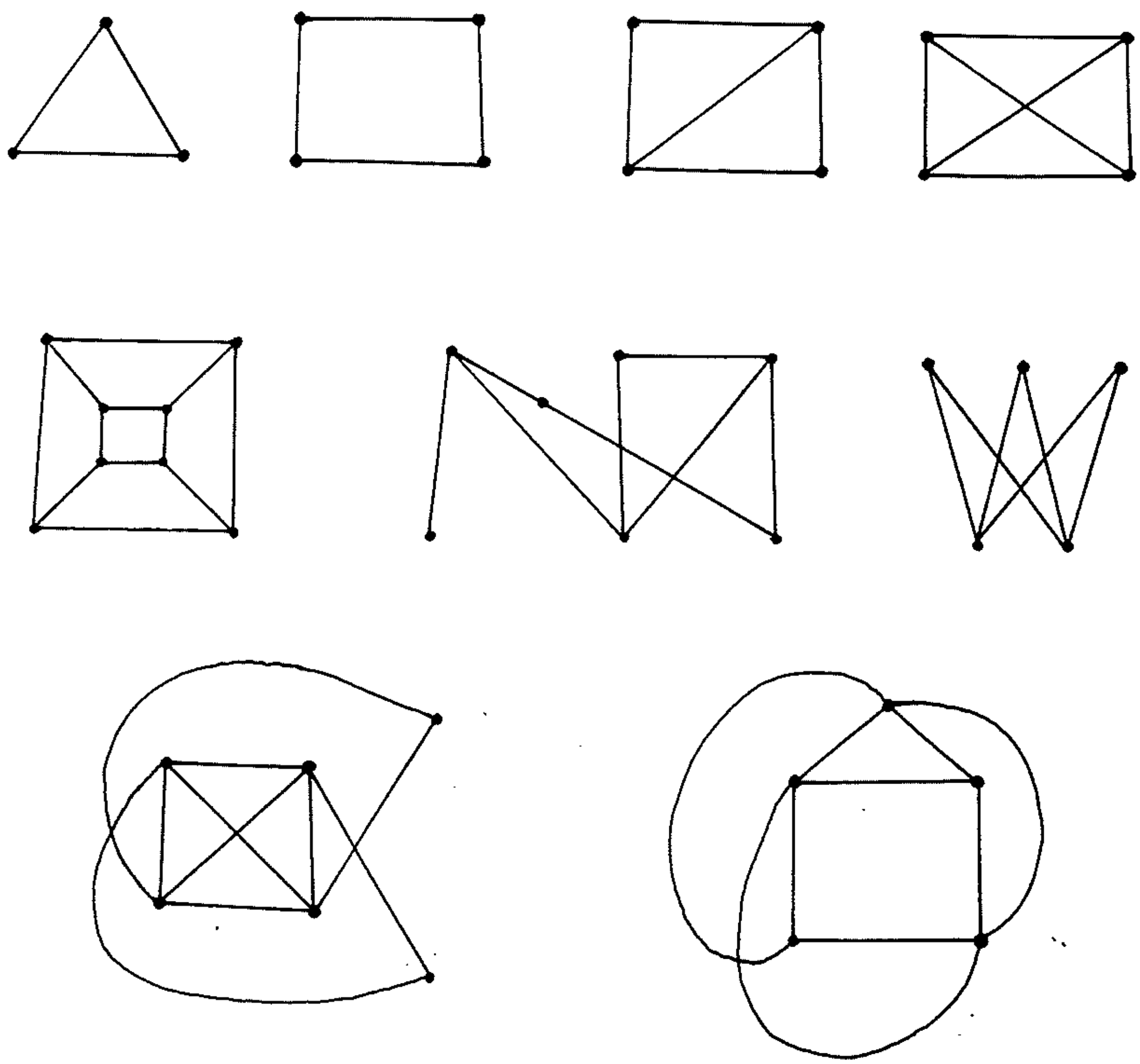
COLORING ACTIVITY #4:
Chromatic Numbers

Definition 1: A coloring of a network assigns to each vertex a color so that no two adjacent vertices share the same color.

Definition 2: The chromatic number of a network is the least number of colors necessary to color the network.

For example, the chromatic number of any triangle is 3, since each vertex must be assigned a unique color.

Determine the chromatic number for each of the following networks.



[DeVino]

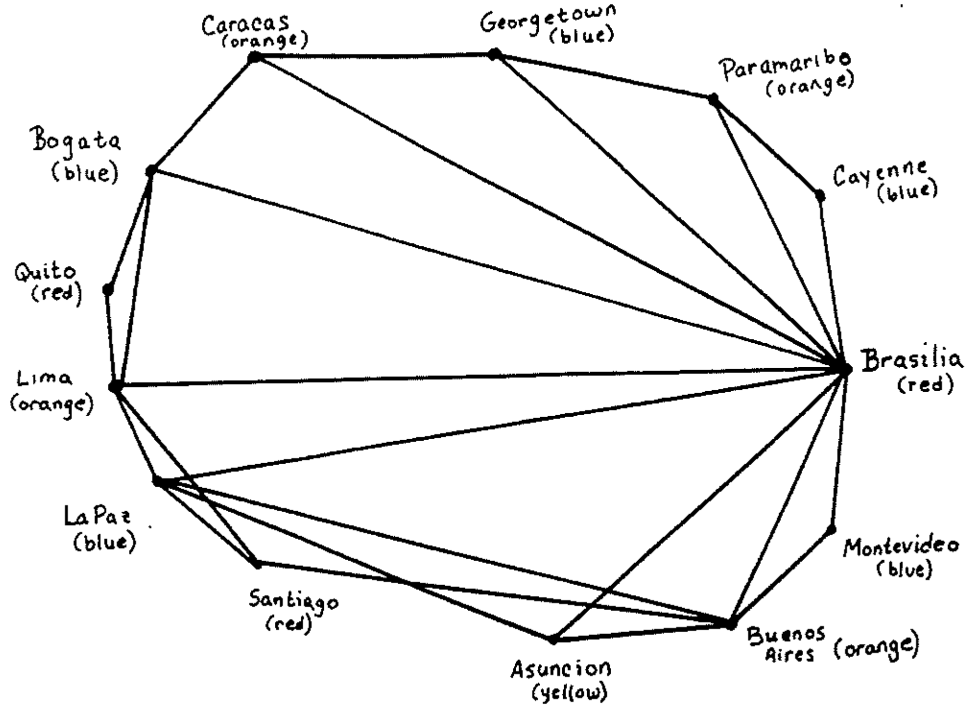
COLORING ACTIVITY #5
Color South America

Put this map into a network, and determine its chromatic number.

Outline Map 9

[Coulter, Simon]

An Answer for the South America Map



4 different colors

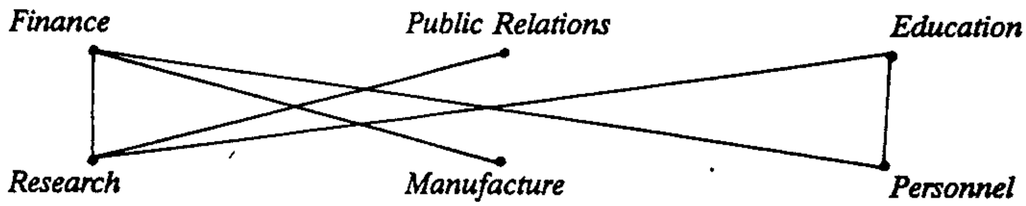
TEACHING NOTES VI:
Using coloring and graphs to solve scheduling problems.

Problem:

A company needs to schedule end-of-the-year meetings for each of its departments: Finance, Personnel, Public Relations, Education, Research, and Manufacture. Some executives work in more than one department, so any two departments with the same members cannot be scheduled on the same day. What is the fewest number of days needed for these meetings?

Ms. Smart is in Finance, Manufacture
Mr. Charisma is in Public Relations, Research
Mr. Biggs is in Education, Research
Mr. Mann is in Personnel, Finance
Ms. Cash is in Finance, Research
Ms. Topps is in Personnel, Education

Hint: use the idea of "different color" for different days.



Above is a graph of the departments with those whose meetings must be scheduled on different days (because they have common members) connected with an edge. The meetings can be scheduled on only 2 days. If Finance is scheduled on Monday, Research, Manufacture, and Personnel cannot meet on Monday, but they can all meet on Tuesday since no two of them have members in common. Public Relations and Education cannot be scheduled on Tuesday, but they can both be scheduled on Monday!

If we replace "days of the week" with "colors", we can say that we "colored" this map.

[Van Hise]

Other applications of chromatic numbers:
1. *Avoid scheduling conflicts.*
2. *Correlation analysis (Copes).*
3. *Binary code transmissions (Copes).*
4. *Chemical storage (Copes).*

Notes: Broadcast Problem, Coloring Activity #7.

The two parts are intended to be used on different days. In part 1, students can struggle to devise their own method of broadcast frequency assignment. Then after they have been given a hint about coloring, let them make a second attempt at solving the problem.

COLORING ACTIVITY #6:
Scheduling Activities.

In order to schedule activities during extra-help/activity periods, have the students write down on the board their names under the clubs or activities that meet during extra help. (Teacher directions: Before the students come into class write on the board the clubs or activities to which the students might belong). Example of clubs and/or activities: social committee, band, orchestra, choir, AFS, Peace Corps, Foreign Language, and honor society. Together make a network. Have the clubs be the vertices and the edges between clubs represent the students belonging to both clubs. Using colors, determine the days the clubs can meet without overlapping participants. The number of different colors will be the number of days needed for each club or activity to meet.

[Coulter, Simon]

COLORING ACTIVITY #7:
"The Broadcast Problem".

The Federal Communication Commission (FCC) regulates the establishment of new radio or television stations in an area. Factors which need to be considered before granting a license to a new station include:

A. The number of existing stations in an area.
B. The transmitting power of each existing station.

Failure to consider the above conditions will likely result in two or more stations interfering with each other's signals.

The Problem:

Suppose you are employed by the FCC. You have received a proposal for the establishment to 10 high frequency stations to be located in each of the following cities:

Albuquerque	Detroit
Atlanta	Los Angeles
Chicago	New Orleans
Dallas	New York
Denver	Seattle

Each station will have the power to broadcast its signal a distance of 500 miles.

* * * * * * * *

PART I

WHAT IS THE MINIMUM NUMBER OF FREQUENCIES (OR CHANNELS) WHICH MUST BE ASSIGNED TO THIS GROUP IN ORDER TO INSURE THAT THERE IS NO STATION INTERFERENCE? (Explain the method used to arrive at your solution.)

NOTE: A map of the United States and a mileage chart is provided to help you.

PART II

Locate the 10 given cities on a US map.

A. Draw segments between cities where the distance is less than 1000 miles. What is the significance of these segments? (Interference possible between cities).

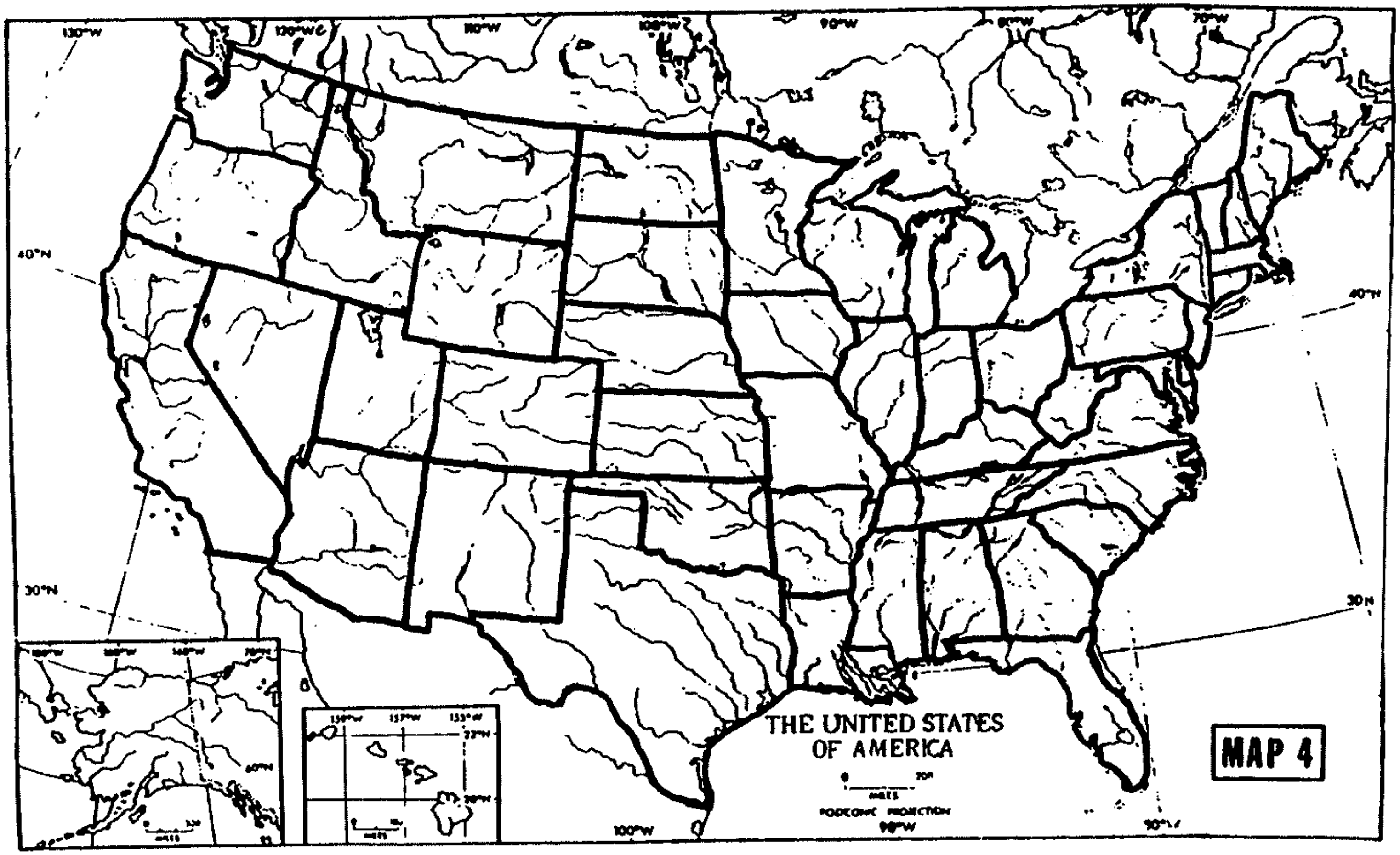
B. Determine the chromatic number for the network produced in part A. What is the significance of the chromatic number? (Equals the minimum number of frequencies in the Broadcast Problem.)

[DeVino]

NAME ______________________________ DATE ______________________________

CLASS ______________________________ TEACHER ______________________________

MAP 4 - UNITED STATES



J. Weston Walch, Publisher Box 658 Portland, Maine 04104

01-7428

MILEAGE CHART

	Alb	Atl.	Chi.	Dal.	Den.	Det.	L.A.	N.O.	N.Y.	Sea.
Albuquerque	XXX	***	***	***	***	***	***	***	***	***
Atlanta	1881	XXX	***	***	***	***	***	***	***	***
Chicago	1281	674	XXX	***	***	***	***	***	***	***
Dallas	638	795	917	XXX	***	***	***	***	***	***
Denver	417	1398	996	781	XXX	***	***	***	***	***
Detroit	1525	699	266	1143	1253	XXX	***	***	***	***
Los Angeles	807	2182	2054	1387	1059	2311	XXX	***	***	***
New Orleans	1134	479	912	496	1273	1045	1883	XXX	***	***
New York	1979	841	802	1552	1771	637	2786	1311	XXX	***
Seattle	1440	2618	2013	2078	1307	2279	1131	2574	2815	XXX

Rand McNally Road Atlas & Travel Guide (1987)

COLORING ACTIVITY #8:
Garbage, Traffic, and Bus Tours.

1. Garbage Problem

There are four garbage collection routes, described below. Each of the four routes is to be assigned to certain days of the week. How can you make this assignment so that no site is visited twice on one day and so that all four tours are completed in as few days as possible? How many days are needed to complete the four tours?

Route 1: Stock exchange, Bronx Zoo, Yankee Stadium.
Route 2: Columbia University, City Hall, Central Park Zoo.
Route 3: Pier 42, East 81st, Bronx Zoo.
Route 4: United Nations, City Hall, Pier 42.

Make a tour graph with the vertices representing the routes and an edge between two routes if a site is visited by the two routes. What is the chromatic number? How many days would it take for these 4 routes to be completed?

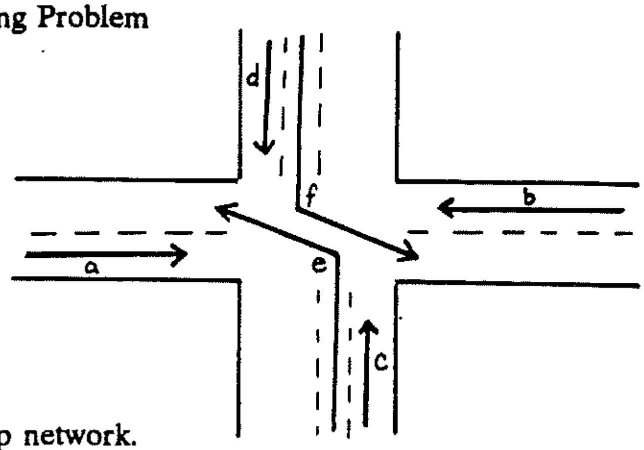
2. Mailpersons

There are four mail delivery routes described below. Each of the four routes is to be assigned to certain days of the week. How can you make this assignment so that no street is visited twice on one day and so that all four tours are completed in as few days as possible? How many days are needed to complete the four tours?

Route 1: 21st to 30th streets
Route 2: 28th to 40th streets
Route 3: 35th to 50th streets
Route 4: 80th to 110th streets

[Coulter, Simon]

3. Traffic Phasing Problem



a) Set up network.
b) Vertices are the traffic streams.
c) Paths are to join two traffic streams by an edge if they are incompatible.
d) The number of different colors will stand for "green" time. How many "green" times are needed?

4. Bus Tours

There are seven bus tours described below. Each of the seven tours is to be assigned to certain days of the week. How can you make this assignment so that no site is visited twice on one day and so that all seven tours are completed in as few days as possible? How many days are needed to complete the seven tours?

Tour 1: Empire State Bldg, Madison Square Garden, 42nd St.
Tour 2: Greenwich Village, 42nd St., Empire State Bldg., Metropolitan Museum.
Tour 3: Shea Stadium, Bronx Zoo, Brooklyn Bridge.
Tour 4: Statue of Liberty, 42nd St.
Tour 5: Statue of Liberty, United Nations, Empire State Bldg.
Tour 6: Shea Stadium, Yankee Stadium, Bronx Zoo.
Tour 7: Columbia University, Bronx Zoo, United Nations.

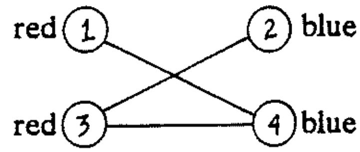
Write up your conclusions:

5. Make up your own problem.

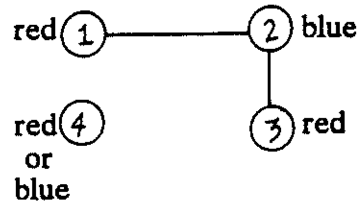
[Coulter, Simon]

Coloring Activity # 8 Solutions

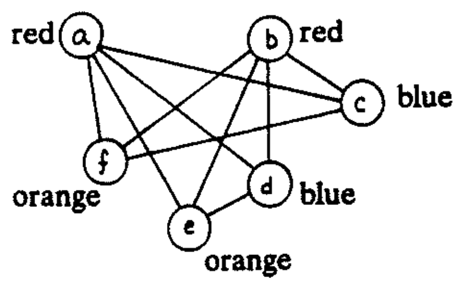
1. Answer: 2 different colors:
2 days.



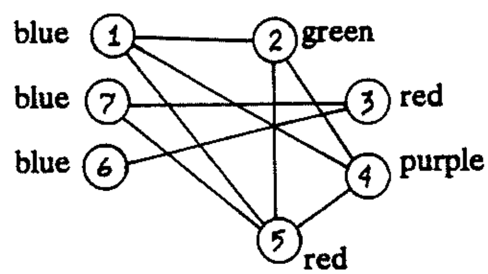
2. Answer: 2 different colors:
chromatic color is
2 and the number of
days is 2.



3. Answer: 3 different colors:
3 "green" times.



4. Answer: 4 colors:
4 days.



Chapter 4: OTHER DISCRETE MATH TOPICS

OTHER DISCRETE MATH TOPICS
TABLE OF CONTENTS

Here is where we put the loose ends -- materials which did not fit into any of the preceding three chapters. However, you will find some interesting materials which might fit into your classrooms.

1: Matrix Multiplication Method for Coding and Decoding

TEACHING NOTES: This material is appropriate for grades 10 - 12, Algebra II and up.

References: *Johnsonbaugh, Richard, Essential Discrete Mathematics, McMillan; New York, N.Y., 1987.*

Seymour Lipschutz, Discrete Mathematics, Schaum's Outline Series, McGraw-Hill; 1976.

Grande, John J., Bisset, W., Barbeau, E., Finite Mathematics, Houghton Mifflin Canada Limited, 1988.

Copes, W., Sloyer, C., Stark, R., and Sacco, W., Graph Theory, Janson Publications, Inc., RI, 1987.

Harshbarger, R. J. and Reynolds, J. J., Mathematical Applications for Management, Life, and Social Sciences, D.C. Heath, Lexington, Mass., 1989.

[Csongor]

Prerequisite skills: Matrix multiplication, finding the inverse of a square matrix.

Background review:

1. Multiplication. Let A and B be two matrices such that the number of columns of A is equal to the number of rows of B, and A is an m x p matrix and B is a p x n matrix. Then the product of A and B, written AB, is the m x n matrix whose ij - entry is obtained by multiplying the elements of the i^{th} row of A by the corresponding elements of the j^{th} column of B and then adding:

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdots & \cdot \\ a_{i1} & \cdots & a_{ip} \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \cdots & \cdot \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$$

2. Inverse of a Square Matrix

a. Method 1 for a 2 x 2 matrix:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad - bc \neq 0$. If $ad - bc = 0$, A^{-1} does not exist.

b. Method 2 for any m x m matrix:

Note: The m x m square matrix with 1's along the main diagonal and 0's elsewhere is called the identity matrix or unit matrix and is denoted by I.

$$\left[\begin{array}{ccc|cccc} a_{11} & \cdots & a_{1m} & 1 & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & 0 & 1 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mm} & 0 & 0 & \cdots & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1m} \\ 0 & 1 & \cdots & 0 & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 & b_{m1} & \cdots & b_{mm} \end{array}\right]$$

1. Form the augmented matrix [A|I] where A is the given m x m matrix and I is the m x m identity matrix.
2. Perform elementary operations on [A|I] until we have an augmented matrix of the form [I|B]; that is, until the matrix A on the left is transformed into the identity matrix.
3. The matrix B on the right side is the inverse of matrix A, that is $B = A^{-1}$.

Lesson:

Given the code:

a	b	c	d	e	f	g	h	i	j	k	l	m	n
1	2	3	4	5	6	7	8	9	10	11	12	13	14

o	p	q	r	s	t	u	v	w	x	y	z	blank
15	16	17	18	19	20	21	22	23	24	25	26	27

and the code matrix:

$$C = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$$

we can encode a message by separating it into pairs of letters, forming 2 x 1 matrices of their code numbers, and multiplying by C. (If the code matrix C were 3 x 3, then we would divide the message by separating it into triplets of letters.) For example the message "I am discrete" is written:

I blank	a m	blank d	i s	c r	e t	e blank
9 27	1 13	27 4	9 19	3 18	5 20	5 27

and encoded by writing each pair of numbers as a column matrix, and multiplying this matrix by the code matrix C.

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ 27 \end{bmatrix} = \begin{bmatrix} 135 \\ 153 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 55 \\ 67 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 27 \\ 4 \end{bmatrix} = \begin{bmatrix} 97 \\ 74 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} 103 \\ 113 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 18 \end{bmatrix} = \begin{bmatrix} 81 \\ 96 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} 95 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 27 \end{bmatrix} = \begin{bmatrix} 123 \\ 145 \end{bmatrix}$$

Thus, 135, 153, 55, 97, 74, 103, 113, 81, 96, 95, 110, 123, and 145 are sent.

To decode the message encoded by C, we find C^{-1} and multiply the codes in similar fashion.

$$C^{-1} = \frac{1}{(3)(5) - (4)(2)} \begin{bmatrix} 5 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix}$$

$$\begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix} \begin{bmatrix} 135 \\ 153 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix} \qquad \begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix} \begin{bmatrix} 55 \\ 67 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

and so on...

Example 2: Using a 3 x 3 code matrix to transmit the message:

$$\underbrace{\overset{M\ \ a\ \ t}{13\ \ 1\ \ 20}}\qquad \underbrace{\overset{h\ \ \text{blank}\ \ i}{8\ \ 27\ \ 9}}\qquad \underbrace{\overset{s\ \ \text{blank}\ \ f}{19\ \ 27\ \ 6}}\qquad \underbrace{\overset{u\ \ n}{21\ \ 14}}$$

$$C = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 2 & -3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 2 & -3 & -2 \end{bmatrix} \times \begin{bmatrix} 13 \\ 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 111 \\ 77 \\ -30 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 2 & -3 & -2 \end{bmatrix} \times \begin{bmatrix} 8 \\ 27 \\ 9 \end{bmatrix} = \begin{bmatrix} 187 \\ 143 \\ -91 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 2 & -3 & -2 \end{bmatrix} \times \begin{bmatrix} 19 \\ 27 \\ 6 \end{bmatrix} = \begin{bmatrix} 197 \\ 145 \\ -74 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 2 & -3 & -2 \end{bmatrix} \times \begin{bmatrix} 21 \\ 14 \\ 27 \end{bmatrix} = \begin{bmatrix} 220 \\ 158 \\ -75 \end{bmatrix}$$

Thus, 111, 77, -30, 187, 143, -91, 197, 145, -74, 220, 158, and -75 are sent.

To decode the message encoded by C, we find C^{-1} (by method 2) and multiply the codes in similar fashion.

$$C^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -5 & 8 & 2 \\ 7 & -11 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 1 \\ -5 & 8 & 2 \\ 7 & -11 & -3 \end{bmatrix} \begin{bmatrix} 111 \\ 77 \\ -30 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ 20 \end{bmatrix}$$

and so on...

<u>Assignments</u>: Students can send their own "secret" messages to each other using a code matrix given by the teacher or their own. It is an interesting sideline to see them make up matrices that are easier to invert. (How can you tell if a matrix has an inverse?)

ACTIVITY 2: The Museum Guard Problem

Teaching Note: This material is appropriate for a geometry class.

I. Background:

" The Museum Guard Problem" was first proposed by Victor Klee at a conference at Stanford in 1973. This problem is also known as "Chvatal's Art Gallery Theorem" or the "Watchmen Theorem".

Imagine yourself running a company that provides security for businesses, schools, museums, etc. Back in the old days, guards could be hired fairly cheaply. However, today's guards not only require better salaries, but also sick pay, vacations, as well as medical and dental insurance.

If you are to stay in business, it is important for you to find a way to guard a building with the least number of guards possible.

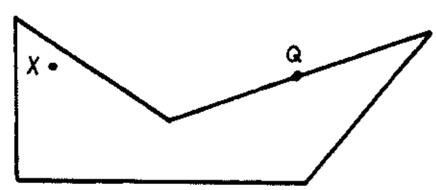
II. The Problem:

You will be given floor plans of various museums. The walls of the museums are covered with valuable paintings. Your museum guards will need to be positioned in such a way that they can see if all of the wall space from a given point in the building. Note: The guards have the ability to look behind them, but they must **remain in one spot without moving.**

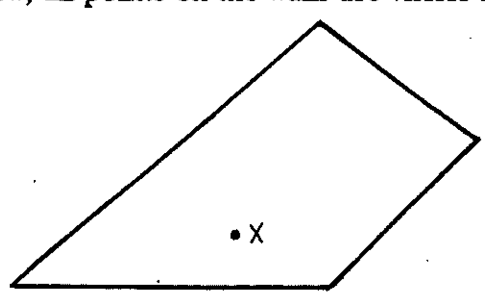
Your objective is to find a pattern between the number of walls in a given museum and the **least** number of guards needed to secure the building.

III. Visibility defined:

In the following museum (polygon), a painting at point Q on the wall is NOT visible from point X.



In the polygon below, all points on the walls are visible from point X.

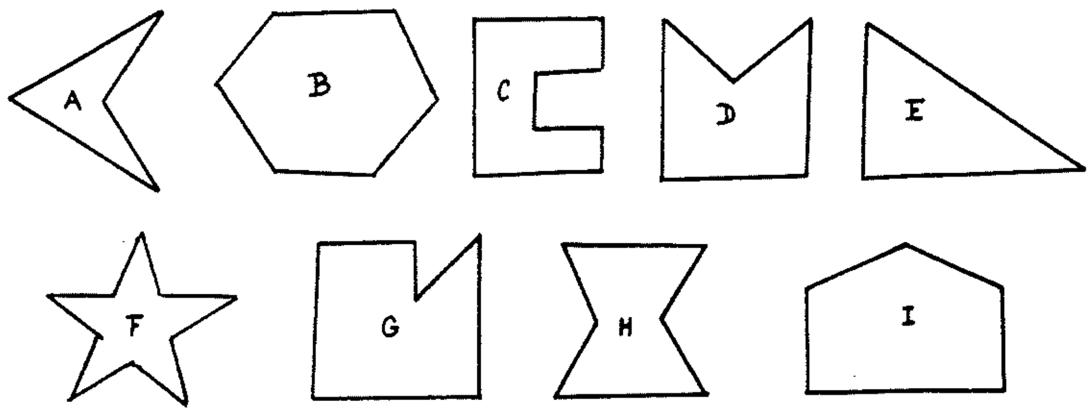


[Devino]

IV. Classifying Museums (Polygons):

Recall that a polygon is a figure in a plane consisting of segments joined at the endpoints. One way of classifying polygons is by counting the number of sides. For example, 3-sided polygons are triangles, 4-sided polygons are quadrilaterals, etc. Polygons can also be classified by looking at their respective angles.

A. Look at the following polygons and find a way to separate them into two general categories by examining their respective angles:



B. Explain your method of classifying the polygons.

C. Separate the polygons according to your classification method.

CATEGORY #1: ______________________________

CATEGORY #2: ______________________________

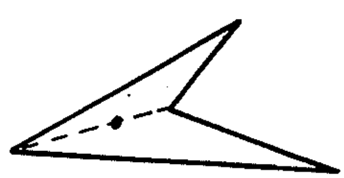
V. Polygons with no angles greater than 180 degrees are called <u>convex</u>.

A. What is the least number of guards needed to watch all the walls for convex polygons?

____

Since all triangles are convex polygons, only one guard is needed to guard a triangular museum.

Any non-convex polygon that can be divided into 2 convex polygons requires only 1 guard provided the guard is positioned along the boundary. For example:



B. Draw a couple of 5-sided non-convex polygons. Draw segment(s) to determine the number of guards needed to see all the walls.

C. State a conclusion about the number of guards needed to see all of the walls in a 5-sided polygon:

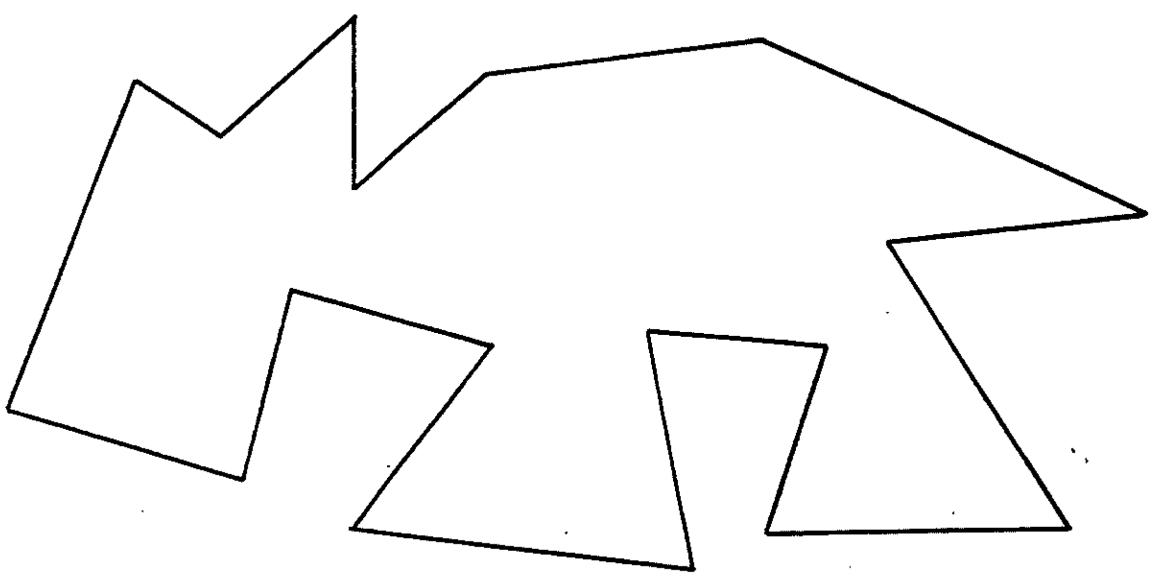
VI. Conjecture: Given a polygon with N sides, the least number of guards needed to watch all the walls is:

$$G = [N/3], \text{ the greatest integer } \leq N/3.$$

The simplest proof by Fisk (1976):

A. Subdivide the polygon into triangles.

B. Show that the vertices of the triangulated polygon can be colored with 3 colors such that no 2 adjoining vertices have the same color.

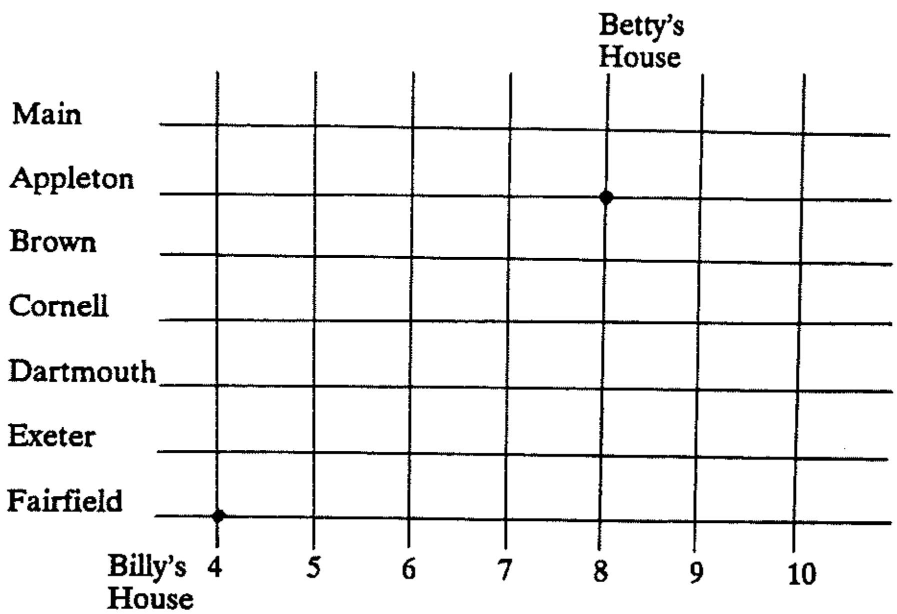


ACTIVITY 3: Paths and Patterns

TEACHING NOTE: A series of lessons counting the number of paths. This material is appropriate for an Algebra II class.

CLASS STARTER: THE BILLY AND BETTY PROBLEM:

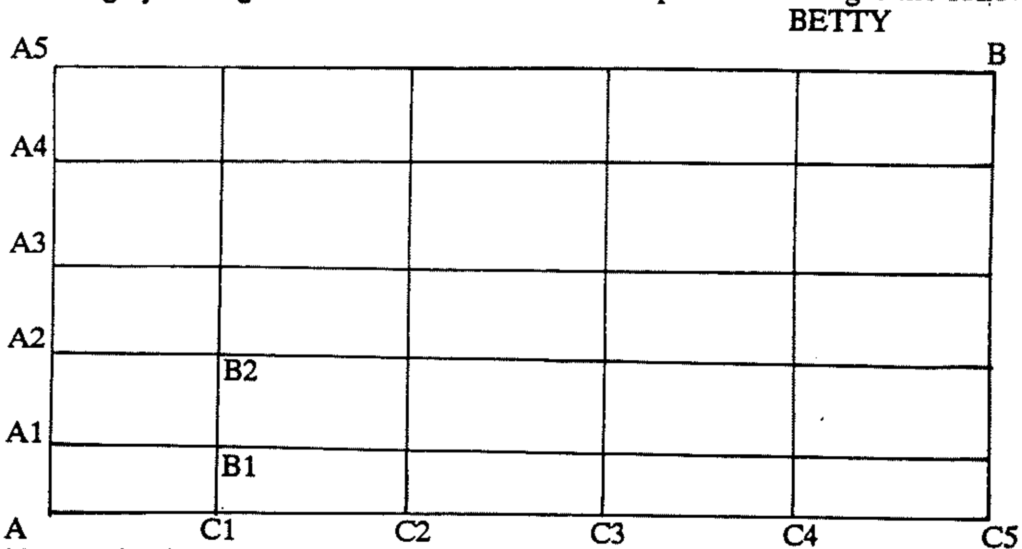
A map of a local town is shown below. Billy lives at the corner of 4th Street and Fairfield Avenue. Betty lives at the corner of 8th Street and Appleton Avenue. Billy decides that he will visit Betty once a day after school until he has tried every different route to her house. Billy agrees to travel only east and north. How many different routes can Billy take to get to Betty's house?



[Johnson]

A SERIES OF LESSONS: <u>Lesson 1:</u>

Let's use the class starter to develop some original mathematical theory. We first organize our thinking by labling the intersections in the above problem. We get the following diagram:



Now we begin small by asking some simpler questions:

How many paths FROM	TO	ANSWER	
A	A	1	
A	A1	1	
A	A2	1	
A	A3	1	
A	C1	1	
A	C2	1	
A	B1	2	(LIST THE PATHS)
A	B2	3	(LIST THE PATHS)

Complete a few more so that the students get the idea.

Change the notation so that we can communicate our thinking: Let's assume we will give coordinates to the grid starting with A(0,0) and using only lattice points.

SOME NOTATION:
<u>Definition:</u> Let the function ALPHA be the number of paths along a grid, moving only north and/or east, from (0,0). That is, for any lattice point B(x,y), such that $x \geq 0$, $y \geq 0$, ALPHA(B) = ALPHA((x,y)) is the number of paths from A(0,0) to B(x,y).

Can we give a formula for finding ALPHA((x,y))?

I. Here is a recursive formula which expresses the number of paths to (x,y) in terms of the number of paths to the points below it and to its left.

$$\text{ALPHA}((x,y)) = \text{ALPHA}((x-1,y)) + \text{ALPHA}((x,y-1))$$

Use the above formula to find ALPHA((3,4)), ALPHA((2,5)).

Lesson 2:

First discuss lesson 1. Then explain the relation between the function ALPHA and Pascal's Triangle.

Introduce new notation that will allow us to compute alpha without referring to a graph or Pascal's Triangle:

DEFINITION: The symbol ! is read "factorial".

$0! = 1$

$1! = 1$

For any whole number n greater than or equal to 2, $n! = n \times (n-1)!$.

CLASS EXAMPLES USING THE DEFINITION:

COMPUTE:

(1) 5! (2) 8! (3) 4!/2! (4) 10!/7!

(5) 12!/(7! x 5!) (5) 8!/(4! x 4!) (7) 5!/(1! x 4!)

Do you notice anything special about the solutions to problems 5 - 7? The numbers are values we have seen in Pascal's Triangle and the numbers can be related to the factorials in the numerator and the denominator.

DEFINITION: Let n be a whole number such that $r \leq n$. We use the two notations nCr and $\binom{n}{r}$ to stand for $\frac{n!}{r!(n-r)!}$.

CLASS EXAMPLES: COMPUTE:

(1) 7C3 (2) $\binom{4}{1}$ (3) $\binom{9}{6}$ (4) 2C2 (5) 14C10

LESSON 2 GROUP WORK

(1) Write the first 12 lines of Pascal's Triangle.

(2) Find ALPHA((3,4)). ____________ Now look for that number in Pascal's Triangle in Row 8. In which positions from the left will you find ALPHA((3,4))?________________________

(3) Find ALPHA((2,3)). ____________ In which row of Pascal's Triangle would we find this value? ______________ In which positions from the left in this row will you find ALPHA ((2,3))? ______________

(4) Give the rows and the positions in Pascal's Triangle you would find the value for ALPHA((4,2)). ____

(5) Use Pascal's Triangle to find the following:

(a) ALPHA((6,3)) (b) ALPHA((7,2))

(c) ALPHA((2,8)) (d) ALPHA((4,6))

(6) Write an algorithm to find ALPHA((a,b)), a $\geq$ 0, b $\geq$ 0, using Pascal's Triangle.

EXTRA CREDIT:

Let BETA((a,b) - (x,y)) be the number of paths from (a,b) to (x,y) along a grid, moving only north or east, where x $\geq$ a, and y $\geq$ b. Write an algorithm using Pascal's Triangle to find the value for BETA.

Lesson 2 Lab NAME:________________

[1] Compute the following:

(A) 0 C 0

(B) 1 C 0   1 C 1

(C) 2 C 0   2 C 1   2 C 2

(D) 3 C 0   3 C 1   3 C 2   3 C 3

(E) 4 C 0   4 C 1   4 C 2   4 C 3   4 C 4

(F) 5 C 0   5 C 1   5 C 2   5 C 3   5 C 4   5 C 5

[2] Explain how the combination notation, nCr, can be used to generate a value in Pascal's Triangle.

[3] Write a formula for ALPHA((a,b)), a $\geq$ 0, b $\geq$ o, using nCr notation.

[4] Write a formula for BETA((a,b) - (x,y)), x $\geq$ a, y $\geq$ b, a $\geq$ 0, b $\geq$ 0, using nCr notation.

[5] Find the following:

(A) ALPHA((10,12))   (B) ALPHA((5,2))

(C) BETA((2,4) - (6,9))   (D) BETA((2,5) - (13,18))

Lesson 3

Discuss the assignment and Lab 2 leading up to the generalization:
ALPHA((a,b)) = (a + b)Ca = (a + b)Cb where a ≥ 0 and b ≥ 0.
Then have the students complete the following Lab 3.

Lesson 3 Lab

On a grid, consider the following points:

A(0,0)
B(3,6)
C(7,2)
D(9,4)
E(13,18)

Find the number of paths along the grid, moving only in a positive direction parallel to the x- or y-axis from:

[1] A to B

[2] A to D

[3] B to D

[4] A to E

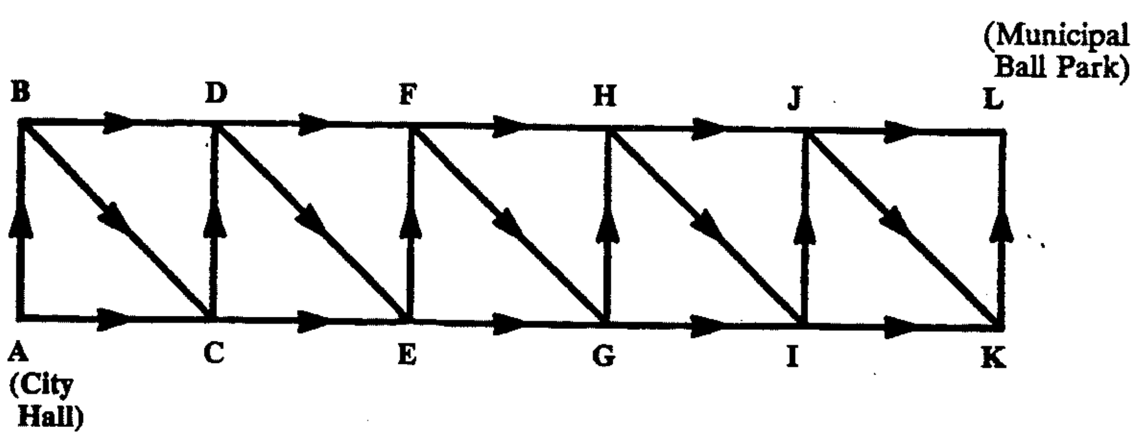
[5] C to E

[6] B to E

[7] B to C

[8] D to E

[9] Here is a map of the streets in a city. All of the streets are one way only as indicated by the arrows. How many different routes are there from City Hall (A) to the Municipal Ball Park (L)?



Note: The following is a follow-up extra credit project given to students after they have completed Lesson 3.

EXTRA CREDIT WORK/RESEARCH

From Lesson 3, problem 9, we obtained the following sequence of numbers:

Routes to:	A	B	C	D	E	F	G	H	I	J	K	L	
	1	1	2	3	5	8	13	21	34	55	89	144	...
term	1	2	3	4	5	6	7	8	9	10	11	12	...

(1) Considering this sequence only, give a rule for generating the n^{th} term.

(2) Find the 15^{th} term of the sequence.

(3) Find the 20^{th} term of the sequence.

(4) What is the special name given to this sequence?

(5) A special sequence can be formed by taking the ratio of consecutive terms. For example, if t_n is the n^{th} term of the sequence a new sequence can be formed as follows:

$$\frac{t_2}{t_1} \quad \frac{t_3}{t_2} \quad \frac{t_4}{t_3} \quad \frac{t_5}{t_4} \quad \cdots \quad \frac{t_n}{t_{n-1}}$$

(a) Write the first eight terms of this sequence.

(b) As n becomes large, the ratio $\frac{t_n}{t_{n-1}}$ becomes what number?

(c) What is the special name given this number in part (b)?

ACTIVITY 4: Tower of Hanoi - How long for 100 disks?

Teaching Note: A description of a lesson on analyzing algorithms. This is appropriate for precalculus students.

The Tower of Hanoi algorithm is deceptively elegant and simple when it is first introduced to students; it is a fine example of the power of recursion.

To move N disks, from tower A to tower B using tower C requires very few statements in Pascal:

```
procedure Hanoi(N:integer; a,b,c:char);
        {precondition: n>=0}
        {move N disks from tower a to tower b using tower c}
        begin
                if N>0 then
                        begin
                                Hanoi(N-1,a,c,b);        {move top N-1 disks from A to temp
                                                          C using B}
                                writeln(a,'----->',b);   {show some action - move last disk to
                                                          B}
                                Hanoi(N-1,c,b,a);        {move top N-1 disks from temp C to
                                                          goal B using A}
                        end;
```

The temptation to run the program with a large number was irresistible to students when they first saw the algorithm. N = 100 was commonly entered. They waited and waited. Nothing happened, except moves continued to be made.

We discussed what was going on, and drew some conclusions about the number of moves for a particular value of N. It didn't take them long to realize that it requires $2^N - 1$ moves. This really didn't sink in for large N. The students decided to let the machine keep running overnight. (I thought that wouldn't hurt the Mac). We put "Do not disturb" signs on the machine, and a note to the curious (i.e., students in other classes) about what was happening. We also decided to enter a counter, so that we could periodically check to see just how much progress had been made. So a global counter (integer) was put in, to be bumped each time Hanoi was called. At some point during the next period, some of the students came back to the classroom, and realized that the counter would soon exceed MAXINT($2^{15} - 1$), and redeclared the counter to be a real number. So far, so good. The next day, the machine was not nearly finished, so we became a bit more scientific.

The students realized that the number of moves for large N is a very large number. (!) We timed the machine on N = 10, with the counter, and discovered that there were 12 moves per second. (The counter slows progress). We calculated that for N = 100, this amounts to about

$$\frac{2^{100} - 1}{12(3600)(24)(365)} \text{ years, or } 3.35(10^{21}) \text{ years.}$$

(Somewhat longer than the expected lifetime of the Macintosh, let alone the teacher or the universe). I then asked them to decide what value of N would cause the machine to run for about 24 hours before it finally finished.

[McIlroy]

Aha! An application of logs.

They computed that in 24 hours, about 1,036,800 moves are possible. What exponent of 2 gets this value? They had to solve the following equation:

$2^N - 1 = 1{,}036{,}800$

$2^N = 1{,}036{,}801$ and taking log of both sides gives:

$\log(2^N) = \log(1{,}036{,}801)$ or

$N = \log(1{,}036{,}801)/\log(2) = 19.98$ or about 20 disks.

With a bit more insight, we set the machine running again, this time with notices pasted on the machine about the number of disks, and the expected stopping time. We had it right, for the machine stopped just short of 24.5 hours after we started it.

Further discussions about exponential complexity and how long it would take for 21 disks, or for 19 disks further hammered home exponential growth. I was pleased to note that they all answered correctly on the next quiz the following question:

What is the order of complexity for the Tower of Hanoi algorithm?

How many moves are required for 10 disks?

If a machine takes 12 hours to move 15 disks, how many hours will it take to move 12 disks?

APPENDIX A:
List of Participants

Name	School	School Address
Randy Beth Appelstein	Bronx High School of Science	Bronx, New York
Valerie Baker	Madison High School	Madison, New Jersey
Barbara J. Chadwick	Pleasantville High School	Pleasantville, New Jersey
Claudia S. Coulter	J. H. Rose High School	Greenville, North Carolina
Julianna E. Csongor	Little Flower High School for Girls	Philadelphia, Pennsylvania
Mary Jane DeBenedictis	Haddon Township High School	Westmont, New Jersey
John DeVino	Colchester High School	Colchester, Vermont
Mary E. Edwards	Lawrence High School	Lawrenceville, New Jersey
Robert E. Eldi	H. B. Thompson High School	Syosset, New York
David T. Farber	Voorhees High School	Glen Gardner, New Jersey
William Hahnenberger	Niskayuna High School	Niskayuna, New York
Robert E. Johnson	West Morris Central High School	Chester, New Jersey
Ray A. Levandowski	North Hunterdon High School	Annandale, New Jersey
Barbara McIlroy	Ridge High School	Basking Ridge, New Jersey
John Pancari	St. Joseph School	Hammonton, New Jersey
Gwen E. Pasquale	Madison High School	Madison, New Jersey
Linda Raymond	Braintree High School	Braintree, Massachusetts
Philip R. Reynolds	Niskayuna High School	Niskayuna, New York
Douglas Schumacher	Syosset High School	Syosset, New York
Joan Shyers	Chatham High School	Chatham, New Jersey
Susan L. Simon	Morristown-Beard School	Morristown, New Jersey
Marguerite Smith	Villa Walsh Academy	Morristown, New Jersey
Patricia A. Van Hise	Hamilton High West	Trenton, New Jersey
Gilbert Waddington	Pennsville Memorial High School	Pennsville, New Jersey
Rebecca Walker	Morris Hills High School	Rockaway, New Jersey
Carolyn Walters	Mount Vernon High School	Mount Vernon, New Jersey
Gisele Zangari	Mansfield High School	Mansfield, Massachusetts

APPENDIX B:
Glossary

Algorithm - A sequence of instructions that solves all instances of a well-defined problem.
Circuit - A path that starts and ends with the same vertex.
Connected Graph - A graph such that for every two vertices A and B, there exists a path from A to B.
Degree of a Vertex - The number of edges incident to the vertex.
Directed Graph - A graph where each edge has a direction on it.
Eulerian Path (Circuit) - A path (circuit) that uses each edge of the graph once and only once.
Graph - A set of points called vertices and a set of lines between some of the vertices, called edges.
Hamiltonian Circuit - A circuit that passes through each vertex once and only once.
Minimum Spanning Tree - A spanning tree with minimum total weight.
Parallel edges - Distinct edges associated with the same pair of vertices.
Path - A sequence of vertices $v_1, v_2, \ldots, v_k$ of a graph such that (v_i,v_{i+1}) is an edge for each i.
Spanning Tree - A subgraph of a graph that is a tree and that contains all of the vertices of the original graph.
Subgraph of a Graph G - A subset of the vertices and edges of G.
Tree - A connected graph with no circuits.
Weighted Graph - A graph in which each edge is assigned a numerical value.

[COZZENS]

APPENDIX C:
References

TEXTS

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APPENDIX D:
Steiner Point Applications and History

This is a summary of the Bern and Graham article cited in Appendix C.

The three point Steiner problem also provides information about the geometry of the shortest Steiner trees.

1) Every angle measures at least 120 degrees. This implies that every given point is connected to the tree by no more than three edges.

2) At every Steiner Point, exactly three edges meet at 120 degree angles.

3) The number of edges in a tree is always one less than the number of given points added to the Steiner Points.

4) Since exactly three edges meet at every Steiner Point and at least one edge must touch every given point, the maximum number of Steiner Points in any problem is two fewer than the number of given points.

For the same number and arrangement of given points, many different Steiner trees can be constructed that have these properties. Some of the trees are known as locally minimal solutions, they cannot be shortened by a small perturbation such as moving an edge slightly or splitting a Steiner Point. But not every locally minimal Steiner tree is a shortest solution. Large scale rearrangements of the Steiner points may be necessary to transform a network into a shortest possible tree, called a globally minimal Steiner tree.

A brute force approach to discovering a shortest network is to search through all possible locally minimal Steiner trees, calculate their lengths and choose the shortest one. Since Steiner points can be placed anywhere, it is not clear that all possible locally minimal Steiner trees can be computed in a finite amount of time.

Z.A. Melzak of the University of British Columbia developed the first algorithm for the Steiner problem. Melzak's algorithm considers many possible connections between given points and many possible locations for the Steiner Points.

1) Separate the set of given points into every possible subset of given points.

2) Create a number of possible Steiner trees for each subset by using a construction similar to the one employed to solve the three-point problem.

After considering all replacement sequences, the algorithm chooses the shortest of these Steiner trees for the subset. Combining the shortest Steiner trees for the subsets in all possible ways to span the original set of given points yields all possible locally minimal Steiner trees and the shortest network can be determined. Melzak's algorithm can take an enormous amount of time even for small problems because it considers so many possibilities. A ten point problem can be separated into 512 subsets of given points. Each of the forty-five subsets of eight points has two million replacement sequences. There exists more that 18,000 ways to recombine the subsets into trees.

There are better ways to organize the computation and increase the speed of the algorithm. Focus on possible patterns of connections in the network; this is called network topology. A topology specifies which points are connected to one another, but not the actual location of the Steiner Points. By assuming a certain topology, an appropriate replacement sequence can be found relatively quickly. For an eight point subset the algorithm needs to consider only about 10,000 different replacement sequences. Since the number of topologies grows rapidly with the size of the subset, Steiner problems become more manageable if only very small subsets of the set of given points are considered.

Computation time of even the most sophisticated algorithm grows exponentially with the number of points. Advances in a theoretical computer science has shown that existing algorithms for Steiner problems cannot be substantially improved. This theory assigns a size to each instance of a problem. For Steiner problems there is a natural measure of size: the number of given points. Next consider the number of basic computer operations such as additions, subtractions, or comparisons an algorithm may need in order to solve an instance of a certain size. Since different instances of the ssame size may require different numbers of operations, the maximum number of operations as a function of size is observed. If the number of operations increase by the size of the instance (n) to some power, as in the expression n^2, 5n, $6n+n^{10}$, the procedure is called a polynomial-time algorithm. If the number of operations increases exponentially with size, as in the expression 2^n, 5^n, or $3n^2 \times 4^n$, the procedure is known as an exponential-time algorithm. For very small problems the solution times of exponential time algorithms are so slow that these algorithms are hopelessly impractical. Even though exponential-time algorithms have been found for Steiner problems such as Melzak's algorithm, no polynomial-time algorithms have yet been found.

Ron Graham, Michael Garey, and David Johnson of AT&T Bell Laboratories probed that the Steiner problem is an NP-hard problem. It is therefore considered unlikely that any NP-hard problem, including the Steiner problem can be solved by a polynomial-time algorithm. Proving that NP-hard problems cannot be solved efficiently is the preeminent problem in theoretical computer science. Edgar Gilbert and Henry Pollock of Bell Laboratories conjectured that the ratio of a shortest Steiner tree is at least 3/2. That is, the Steiner tree's length is at most 13.4% shorter than the minimum spanning tree's length. The 3/2 ratio occurs in the simple example of three points in an equilateral triangle. Chung and Ron Graham have proved that the Steiner tree is at most 17.6% shorter than the spanning tree.

The minimum spanning tree and the shortest-network problems have been applied to constructing telephone pipelines and roadway networks. The solutions provide guidelines for the layout of the networks and the necessary amounts of materials. The most practical application of the Steiner problem is in the design of electronic circuits. A short network of wires on an integrated circuit requires less time to charge and discharge than a long network and so increases the circuit's speed of operation. The most surprising application of the Steiner problem is in the area of phylogeny. David Saukoff of the University of Montreal defined a version of the Steiner problem in order to compute plausible phylogenetic trees. The workers first isolate a particular protein that is common to the organism they want to classify. For each organism, they then determine the sequence of amino acids that make up the protein, and define a point to be a position determined by the number of differences between the corresponding organism's protein and the protein of the other organisms. In a shortest network for this abstract arrangement of given points, the Steiner Points correspond to the most plausible ancestors and the edges correspond to a relation between organism and ancestor that assumes the fewest mutations.

Although knowledge about algorithms has progressed greatly, the shortest network problem remains difficult. A tiny variation in the geometry of a problem may appear to be insignificant, yet it can be a radical alteration of the shortest network for the problem. The shortest network problem is still very challenging.

APPENDIX E
Constructing Models for Minimum Networks Using Soap Bubbles

Materials
Plexiglass (1/8") - 30" by 36".
Plexiglass knife.
Brass bolts (1") - 30 (try to make them very narrow).
Brass nuts - 90 (three for each bolt).
Ruler.
Screwdriver.
Drill (with bit diameter = bolt diameter).
Sandpaper (220 grit).

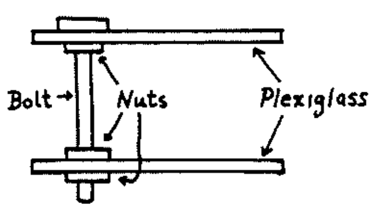
These models are designed to be shown on an overhead projector. The dimensions are approximate and should work well with most overhead projectors. These models can be constructed in a short time and will last for years. If you do not want to construct them you can often find a willing student.

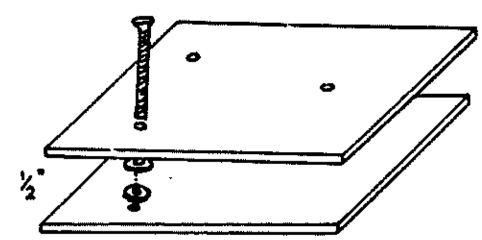
STEP 1: Cut the plexiglass into 5" x 6" rectangles. Plexiglass can be cut by scoring the surface repeatedly (three or more times) with a plexiglass knife. The sheet should be placed on a flat table and strapped along the scored line. Sand the edges lightly.

STEP 2: You should have several blank pieces of plexiglass. You can make several models and the method for each one is the same. First, mark the points you want to drill. The easiest to make first is a triangle. Drill the pieces of plexiglass by placing one on top of another. Drill through both pieces at the same time. It usually helps to place a bolt through the hole to prevent the pieces from shifting. Continue to drill until all three holes are drilled.

STEP 3: Separate the plexiglass pieces. Insert bolts through the holes of one and then secure them with a nut. Screw another nut on each bolt. Measure to be sure that the second nut is 1/2" from the base.

STEP 4: Place the second piece of plexiglass on the first. It will rest on the nuts and should be 1/2" above the other piece. Secure this piece in place with the third bolt.





You can make several models by repeating the steps. Suggested models are a triangle, a square, a pentagon, a hexagon, a map of your state with important cities networked, and randomly drilled holes that will allow you to experiment.

When using the models, you will find that the soap solution adheres to the bolts. If you gently blow the soap the minimum surfaces will form. With a little creativity these models can be used in many ways. There are many nice applications with three dimensional models.

[LAMB]