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Generating-tree isomorphisms for pattern-avoiding involutions*

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Abstract

We show that for $k \geq 5$ and the permutations $\tau_k = (k-1)k(k-2)\ldots 312$ and $J_k = k(k-1)\ldots 21$, the generating tree for involutions avoiding the pattern τ_k is isomorphic to the generating tree for involutions avoiding the pattern J_k. This implies a family of Wilf equivalences for pattern avoidance by involutions.

1 Introduction

1.1 Overview of results

Here we prove a theorem that implies affirmative answers to a conjecture and a question of the first author [Jag03] about pattern avoidance by involutions. Analogous properties are known to hold for pattern avoidance by permutations (without the restriction to involutions), so from a broader perspective this work shows an additional way in which the involutions in the symmetric group parallel the symmetric group as a whole.

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The *pattern* of a word $w = w_1w_2 \dots w_k$ which contains $j \le k$ distinct letters is the order-preserving relabeling of w with $[j] = \{1, 2, \dots, j\}$. Given a word $\pi = \pi_1\pi_2 \dots \pi_n$ we say that π *avoids* the pattern $\sigma = \sigma_1\sigma_2 \dots \sigma_k$ if it has no subsequence $\pi_{i_1} \dots \pi_{i_k}$, $i_1 < \dots < i_k$, whose pattern is σ. Typically, π is a permutation in the symmetric group $\mathcal{S}_n$ and σ a permutation in $\mathcal{S}_k$; here, our focus is on the case when π is an involution in $\mathcal{S}_n$, *i.e.*, when π^2 is the identity permutation.

Let $\mathcal{I}_n$ denote the set of involutions in $\mathcal{S}_n$, let $\mathcal{I}_n(\sigma)$ (respectively, $\mathcal{S}_n(\sigma)$) denote the set of involutions (respectively, permutations) in $\mathcal{S}_n$ that avoid the pattern σ, and let $I_n(\sigma) = |\mathcal{I}_n(\sigma)|$ (respectively, $S_n(\sigma) = |\mathcal{S}_n(\sigma)|$). We write $\sigma \sim_{\mathcal{I}_n} \alpha$ (respectively, $\sigma \sim_{\mathcal{S}_n} \alpha$) if, for every n, $I_n(\sigma) = I_n(\alpha)$ (respectively, $S_n(\sigma) = S_n(\alpha)$); in this case we also say that σ and α are *$\mathcal{I}_n$-Wilf-equivalent* (respectively, *$\mathcal{S}_n$-Wilf-equivalent* or simply *Wilf-equivalent*).

For $k \ge 5$, let $J_k = k(k-1) \dots 321 \in \mathcal{S}_k$ and let $\tau_k = (k-1)k(k-2) \dots 312 \in \mathcal{S}_k$. Here we show that, for every $k \ge 5$, the *generating tree* (whose definition we recall below) for the J_k-avoiding involutions is isomorphic to the generating tree for the τ_k-avoiding involutions. As a corollary, we have that $J_k \sim_{\mathcal{I}_n} \tau_k$ for every $k \ge 5$. (The work in this paper was first reported in the talk [JM07]; a nice generalization of this corollary was subsequently proved independently using another approach by Dukes, Jelínek, Mansour, and Reifegerste in [DJMR09].) In addition to finishing the classification of $\mathcal{S}_5$ according to $\mathcal{I}_n$-Wilf equivalence, this parallels the previously known [Wes90] Wilf equivalence $J_k \sim_{\mathcal{S}_n} \tau_k$ (which follows from techniques that do not apply to involutions).

We provide some relevant background in the rest of this section, prove our main results in Section 2, and then close with applications to enumeration questions and a discussion or related open problems in Section 3.

1.2 Background

The graphs of involutions are characterized by a certain symmetry; plotting π_i as a function of i yields a graph that is symmetric about the line $\pi_i = i$ iff π is an involution. The symmetries of the square that respect this symmetry are generated by the inverse $\pi \mapsto \pi^{-1}$ and the reverse complement $\pi \mapsto \pi^{rc}$. (For $\pi = \pi_1 \dots \pi_n \in \mathcal{S}_n$, $\pi^{rc} = (n+1-\pi_n) \dots (n+1-\pi_1)$.) If ρ is one of these four symmetries, then $\pi \in \mathcal{I}_n$ contains the pattern σ iff $\rho(\pi)$ contains the pattern $\rho(\sigma)$; thus, $\sigma \sim_{\mathcal{I}_n} \rho(\sigma)$ trivially. The $\mathcal{I}_n$-Wilf-equivalences that do not arise from these symmetries are then of particular interest. Because of the symmetry that characterizes the graphs of involutions, moving from Wilf- to $\mathcal{I}_n$-Wilf-equivalence is the same as imposing a symmetry condition on traditional pattern-avoidance questions.

General work on pattern avoidance by involutions includes the seminal paper by Simion and Schmidt [SS85], who determined the $\sim_{\mathcal{I}_n}$-classes of $\mathcal{S}_3$ and gave formulae for $I_n(\tau)$ for $\tau \in \mathcal{S}_3$. Other enumerative results have been obtained by Regev [Reg81] ($I_n(1234)$ and asymptotics for $I_n(12 \dots k)$), Gouyou-Beauchamps [GB89] ($I_n(12345)$ and $I_n(123456)$), and Gessel [Ges90] (determinantal formulas for $I_n(12 \dots k)$). Guibert [Gui95] studied involutions avoiding patterns of length 4, proving the equivalences $3412 \sim_{\mathcal{I}_n} 4321$ and $2143 \sim_{\mathcal{I}_n} 1243$; Guibert, Pergola, and Pinzani [GPP01] subsequently proved $2143 \sim_{\mathcal{I}_n}$

1234. Bousquet-Mélou and Steingrímsson proved the following general result.

Theorem 1.1 ([BMS05])**.** *Let $\sigma_{j+1}\sigma_{j+2}\dots\sigma_k$ be a permutation of $[k]\setminus[j]$. Then*

$$12\dots j\sigma_{j+1}\sigma_{j+2}\dots\sigma_k \sim_{\mathcal{I}_n} j\dots 21\sigma_{j+1}\sigma_{j+2}\dots\sigma_k.$$

This answered a conjecture of the first author [Jag03], who had proved the $j=2,3$ cases of this theorem in finishing the classification of $\mathcal{S}_4$ according to $\mathcal{I}_n$-Wilf-equivalence. More recently, Krattenthaler [Kra06] gave an alternative approach to this (and other results) using growth diagrams.

In [BM03], Bousquet-Mélou studied a functional equation connected to the tree $\mathcal{T}(J_5)$; the construction used there is generalized to $\mathcal{T}(J_k)$ here.

There is a large body of literature on Wilf-equivalence; a particular result that is of interest to us is the following theorem, due to West.

Theorem 1.2 ([Wes90])**.** *For $k\geq 3$, let $\sigma_3\sigma_4\dots\sigma_k$ be an arbitrary permutation of $[k]\setminus[2]$. Then*

$$12\sigma_3\sigma_4\dots\sigma_k \sim_{\mathcal{S}_n} 21\sigma_3\sigma_4\dots\sigma_k.$$

Invoking this theorem as well as symmetry operations that preserve Wilf-equivalence, we immediately have the following corollary.

Corollary 1.3. *For every $k\geq 5$,*

$$J_k \sim_{\mathcal{S}_n} \tau_k.$$

Because the symmetry operations used to prove Corollary 1.3 do not preserve the involutive nature of permutations, the analogous result for $\mathcal{I}_n$-Wilf-equivalence does not follow from the $j=2$ case of Theorem 1.1. In [Jag03], the first author conjectured that $J_5 \sim_{\mathcal{I}_n} \tau_5$ and asked whether this might be an instance of a more general theorem. Our Corollary 2.17 below gives such a general theorem; this is the $\sim_{\mathcal{I}_n}$-analogue of Corollary 1.3 and so may be viewed as establishing another parallel between $\mathcal{S}_n$ and $\mathcal{I}_n$.

As observed in [Jag03], the conjecture later proved by Bousquet-Mélou and Steingrímsson would not account for all of the conjectured $\mathcal{I}_n$-Wilf-equivalences among patterns of length 5; in particular, numerical results suggested that $12345 \sim_{\mathcal{I}_n} 45312$. Furthermore, $123456 \not\sim_{\mathcal{I}_n} 453126$, so the prefix 12345 cannot in general be replaced by the prefix 45312 to obtain a $\mathcal{I}_n$-Wilf-equivalent pattern and a different technique is needed to prove this conjectured equivalence.

2 A family of generating-tree isomorphisms

2.1 The trees

Definition 2.1. For a pattern $\tau\in\mathcal{S}_k$, let $\mathcal{T}(\tau)$ be the tree of τ-avoiding involutions in which $\pi\in\mathcal{I}_n(\tau)$ is a child of $\sigma\in\mathcal{I}_{n'}(\tau)$ (with $n'=n-2$ or $n'=n-1$) iff deleting the cycle containing n from π produces a permutation whose pattern is σ. (In [Gui95], this is the construction of a tree using the "recurrence method.") We will include the empty involution $\epsilon \in \mathcal{S}_0$ (whose graph is a 0×0 square) in this tree. Observe that the parent in this tree of a non-empty involution $\pi \in \mathcal{S}_n$ is an involution in $\mathcal{S}_{n-2}$ if $\pi(n) \neq n$ and an involution in $\mathcal{S}_{n-1}$ if $\pi(n) = n$, and that this construction does indeed yield a tree whose root is ϵ.

We now define the trees that we will prove are isomorphic to the generating trees of J_k and τ_k. There is a slight dependence on the parity of k.

Definition 2.2 (The k-tree)**.** The *k-tree* is defined as follows. Let $a = \frac{k}{2} - 1$ if k is even and $\frac{k-1}{2}$ if k is odd, and label the root node of the tree with $(n, y_1, \ldots, y_a) = (0, 0, \ldots, 0, 0)$. A node with label $(n, y_1, \ldots, y_a)$ has $y_a + 2$ children; the set of labels of these nodes is:

$$\{(n+1, q, y_2+1, \ldots, y_a+1)\} \cup \bigcup_{j=0}^{y_a} \{(n+2, z_1(j), \ldots, z_a(j))\}, \tag{1}$$

where, in the label whose first component is $n+1$,

$$q = \begin{cases} y_1 + 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases},$$

and in the other labels

$$z_i(j) = \begin{cases} y_i + 2 & j \leq y_{i-1} \\ j+1 & y_{i-1} < j \leq y_i \quad (j \leq y_i \text{ for } i = 1) \\ y_i + 1 & y_i < j \end{cases}.$$

Note that for a label $(n, y_1, \ldots, y_a)$, for $0 < i < a$ we have $0 \leq y_i \leq y_{i+1} \leq n$. We also have $y_i < y_{i+1}$ unless $y_i = y_{i+1} = n$.

We say that labels of the form $(n, y_1, \ldots, y_a)$, and their corresponding nodes, form *level n of the k-tree*. Thus the node (at level n) with label $(n, y_1, \ldots, y_a)$ has 1 child at level $n+1$ and $y_a + 1$ children at level $n+2$.

The first 5 levels of the 5-tree, with nodes labeled as in Definition 2.2, are shown in Figure 1. This illustrates the remark above that the number of children on level $n+2$ of a node with label $(n, \ldots, y_a)$ equals $y_a + 1$. We also observe that, in Figure 1, the number of children on level $n+2$ of a node with label (n, y_1, y_2) that have the maximum number of children on level $n+4$ (*i.e.*, whose labels are of the form $(n+2, y_1', y_2+2)$) equals $y_1 + 1$. (In fact, it was this observation about $\mathcal{T}(\tau_k)$ that originally led to the labels that we present here.) Considering Equation (1) we see that an analogous result holds in general.

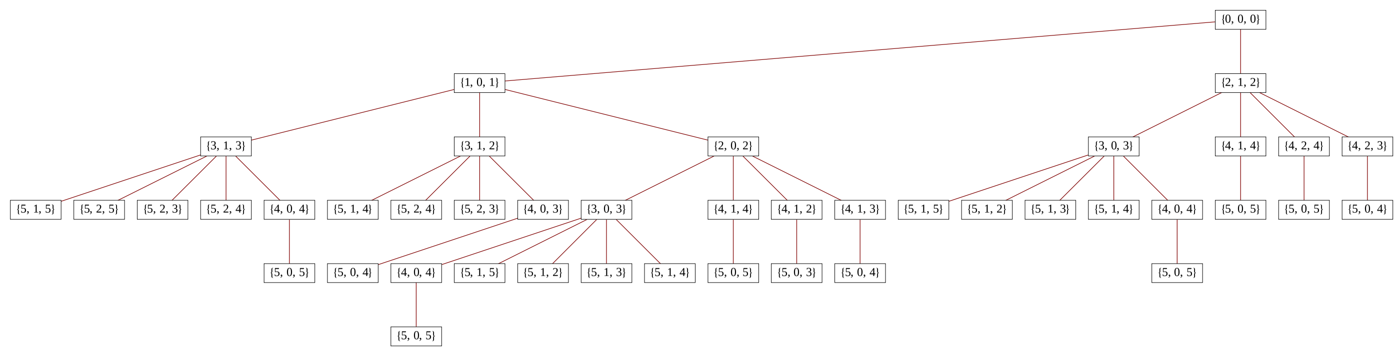

Figure 1: Levels 0–5 of the 5-tree with nodes labeled as in Equation (1).

2.2 Involutions avoiding $J_k = k(k-1)\dots 321$

Theorem 2.3. *For $k \geq 5$, the generating tree for involutions avoiding $J_k = k(k-1)\dots 21$ is isomorphic to the k-tree of Def. 2.2.*

Definition 2.4. Given a permutation $\pi \in \mathcal{S}_n$, we say that we *add the cycle* $(m(n+2))$ *to* π *in order to obtain* σ if $\sigma \in \mathcal{S}_{n+2}$ is a permutation that contains the cycle $(m(n+2))$ and if deleting m and $n+2$ from the list of values of σ produces a word whose pattern is π. Alternatively, σ is constructed from π by adding 1 to all values of π that are at least m, inserting $n+2$ just before m^{th} position in π, and appending m to the end.

Before proving Thm. 2.3, we need the following lemma.

Lemma 2.5. *For $\pi \in \mathcal{I}_n(J_k)$, let $\sigma \in \mathcal{I}_{n+2}$ be obtained by adding the cycle $(m(n+2))$ to π. For $l \geq 2$ and $i > n+2-m$, if the $i \times i$ box in the upper-right corner of the graph of σ contains J_l, then either (1) there is a J_l in the $(i-2)\times(i-2)$ box in the upper-right corner of the graph of π or (2) the box in σ contains a J_l that involves both of the dots added to π to create σ.*

Proof. A J_l in the $i \times i$ box of σ that does not involve either of the new dots corresponds (after deleting the newly-added rows and columns) to a J_l in the $(i-2)\times(i-2)$ box in π. Given a J_l in σ that contains only one of the newly-added dots, by symmetry we may assume the new dot is in the top row. For even l, if $l/2$ of the dots that form this J_l are strictly above the diagonal ($\sigma_i > i$) or, for odd l, if $\lceil l/2 \rceil$ of these dots are weakly above the diagonal ($\sigma_i \geq i$), then these dots and their reflections across this diagonal form a decreasing sequence of length at least l that involves both of the new dots as required. Otherwise, the other dots from the J_l and the reflections of those dots together form a decreasing sequence of length at least l that involves neither of the new dots (and so produces a J_l in π as required). □

We may now turn to the proof of Thm. 2.3.

Proof of Thm. 2.3. Given $\pi \in \mathcal{I}_n(J_k)$, we first define the label of π and then describe the labels of the children of π in the tree $\mathcal{T}(J_k)$. For the labels of π:

k **even** Let p_i be the side length of the largest square in the upper-right corner of the graph of π that avoids the decreasing sequence of length $2i$ for $1 \leq i \leq a = \frac{k}{2} - 1$. Assign to π the label $(n, p_1, \dots, p_a)$.

k **odd** Let p_i be the side length of the largest square in the upper-right corner of the graph of π that avoids the decreasing sequence of length $2i-1$ for $1 \leq i \leq a = \frac{k-1}{2}$. Assign to π the label $(n, p_1, \dots, p_a)$.

In both of these cases, the empty permutation ϵ has label $(0, 0, \dots, 0)$. We now describe the multiset of labels of the children of an involution $\pi \in \mathcal{I}_n(J_k)$ whose label is $(n, p_1, \dots, p_a)$ and show that this is described by Equation (1).

If we add $n+1$ to π as a fixed point, the resulting permutation is an involution in $\mathcal{S}_{n+1}$ that again avoids J_k. The added dot can be part of a J_1—indeed, it forms a J_1 in the top-right corner of the resulting graph—but it cannot be part of any J_i for $i > 1$. Thus, the label of this involution is $(n+1, p_1+1, \ldots, p_a+1)$ if k is even and $(n+1, 0, p_2+1, p_3+1, \ldots, p_a+1)$ if k is odd. This accounts for the label at level $n+1$ in Equation (1).

If we add the cycle $(m(n+2))$ to π to obtain σ, then the $i \times i$ box in the upper-right corner of the graph of σ does not contain a J_l iff either of the following two conditions holds:

1. The new dots are outside of this box (*i.e.*, $m \leq n+2-i$) and the $(i-1) \times (i-1)$ box in the upper-right corner of π avoided J_l.

2. The new dots are inside of this box (*i.e.*, $m > n+2-i$), $l > 2$, the $(n+1-m) \times (n+1-m)$ box in the upper-right corner of π avoided J_{l-2}, and the $(i-2) \times (i-2)$ box in the upper-right corner of π avoided J_l. (By Lemma 2.5, if either of these added dots is part of a J_l in this $i \times i$ box then both of them are, so this condition is indeed sufficient.)

If we add the cycle $(m(n+2))$ to $\pi \in \mathcal{I}_n(J_k)$ in order to obtain $\sigma \in \mathcal{I}_{n+2}(J_k)$ then the $(n+2) \times (n+2)$ box in the upper-right corner of the graph of σ (*i.e.*, the entire graph of σ) avoids J_k; because taking $i = n+2$ makes the first condition above unsatisfiable, the second condition must be satisfied. In particular, the $(n+1-m) \times (n+1-m)$ box in the upper-right corner of π must avoid J_{k-2}, but this is the only restriction on our choice of m. If we let $j = n+1-m$, this restriction becomes $0 \leq j \leq p_a$; these various choices of j give all of the children of π that are level $n+2$ of $\mathcal{T}(J_k)$.

We now determine the label $(n+2, q_1, \ldots, q_a)$ of σ for the possible choices of j. If $i > 1$ and $j \leq p_{i-1}$ then the added dots cannot take part in any J_{2i} (if k is even) or J_{2i-1} (if k is odd). The largest upper-right square avoiding this pattern (J_{2i} or J_{2i-1} as appropriate) in σ is thus formed from the corresponding square in π along with the two added rows/columns, so $q_i = p_i + 2$.

If $j \leq p_1$, the $(j+2) \times (j+2)$ box in the upper-right of σ contains the new dots, which form a J_2 (and thus a J_1); because $j \leq p_1$, the $(j+1) \times (j+1)$ box avoids the appropriate pattern, so $q_1 = j+1$. For $i > 1$ and $p_{i-1} < j \leq p_i$, there is a J_{2i-2} or J_{2i-3} (as appropriate) in the upper-right $j \times j$ box of π because $j > p_{i-1}$; the added dots combine with this to produce a J_{2i} or J_{2i-1}, so $q_i \leq j+1$. Because $j \leq p_i$, the $(j+1) \times (j+1)$ box in the upper-right corner of σ does not inherit a J_{2i} or J_{2i-1} from π, so in fact $q_i = j+1$.

Finally, if $p_i < j$ then the added dots are outside of a box that already contains J_{2i} or J_{2i-1} as part of π. The newly added top row and rightmost column enlarge the box that avoids the appropriate pattern by 1, so $q_i = p_i + 1$.

Thus, as we let j range over all allowed values, we obtain exactly the level-$(n+2)$ labels in Equation (1). Adding the label of the level-$(n+1)$ child discussed above finishes the proof. $\square$

2.3 Involutions avoiding $\tau_k = (k-1)k(k-2)\dots 312$

Theorem 2.6. *For $k \geq 5$, the generating tree for involutions avoiding $\tau_k = (k-1)k(k-2)\dots 312$ is isomorphic to the k-tree of Def. 2.2.*

We start with the labeling that we will use for the involutions in the tree of Thm. 2.6; these labels will be used in some of the lemmas needed for the proof of Thm. 2.6 below.

Definition 2.7. For $\pi \in \mathcal{I}_n(\tau_k)$, we define its label as follows (depending on the parity of k):

k **even** Let p_i be the side length of the largest square in the upper-right corner of the graph of π that avoids the decreasing sequence of length $2i$ for $1 \leq i < a = \frac{k}{2} - 1$. Let p_a be the number of children of π that are on level $n+2$ of $\mathcal{T}(\tau_k)$. Assign to π the label $(n, p_1, \dots, p_a)$.

k **odd** Let p_i be the side length of the largest square in the upper-right corner of the graph of π that avoids the decreasing sequence of length $2i-1$ for $1 \leq i < a = \frac{k-1}{2}$. Let p_a be the number of children of π that are on level $n+2$ of $\mathcal{T}(\tau_k)$. Assign to π the label $(n, p_1, \dots, p_a)$.

In both of these cases, the empty permutation ϵ has label $(0, 0, \dots, 0, 1)$; note that we may always add the 2-cycle $((n+1)(n+2))$ to π in order to obtain a child of π at level $n+2$ of $\mathcal{T}(\tau_k)$, so $p_a \geq 1$.

We will also use the following definition.

Definition 2.8. For $\pi \in \mathcal{I}_n(\tau_k)$, we say that j is an *active value of* π if, for $m = n+1-j$, we may add the cycle $(m(n+2))$ to π (in the sense of Definition 2.4) in order to obtain $\sigma \in \mathcal{I}_n(\tau_k)$. We also say that we *use the active value j to obtain σ* if σ is the involution constructed from π in this way.

We start with a number of lemmas that we will need for the proof of Thm. 2.6 before turning to that proof.

Lemma 2.9. *If adding a cycle $(m(n+2))$ to $\pi \in \mathcal{I}_n(\tau_k)$ produces $\sigma \in \mathcal{I}_{n+2}$ that contains τ_k, then the value $n+2$ is included in some instance of a τ_k in σ.*

Proof. Because π did not contain τ_k, at least one of the values $n+2$ and m must participate in an instance of τ_k. Because σ is an involution, it contains the reflection (across the line of symmetry defining the graphs of involutions) of any τ_k involving the value m (which occurs in the $(n+2)^{\text{nd}}$ column); this is again an instance of τ_k, and it involves the value $n+2$ (in the m^{th} column). $\square$

Lemma 2.10. *If adding a cycle $(m(n+2))$ to $\pi \in \mathcal{I}_n(\tau_k)$ produces $\sigma \in \mathcal{I}_{n+2}$ that contains an instance of τ_k, then σ contains an instance of τ_k that involves both m and $n+2$ (the values in the rightmost column and the top row, respectively, of σ).*

Proof. By Lemma 2.9, σ contains an instance of τ_k that involves the value $n+2$. For even k, if $k/2$ of these dots are strictly above the diagonal ($\sigma_i > i$) or, for odd k, if $\lceil k/2 \rceil$ of these dots are weakly above the diagonal ($\sigma_i \geq i$), then the reflection of these dots across the diagonal gives a set of dots below the diagonal which are also part of the graph of σ (by the symmetry of involutions) and include the dot in the rightmost column of σ. The union of these two sets of dots contains a τ_k involving both the values m and $n+2$. □

Lemma 2.11. *Fix $\pi \in \mathcal{I}_n(\tau_k)$ with label $(n, p_1, \ldots, p_a)$, let σ be obtained from π by using the active value $j \leq p_{a-1}$, and let $(n+2, q_1, \ldots, q_a)$ be the label of σ. Then for all $j' > q_{a-1}$, j' is an active value of σ iff $j' - 2$ is an active value of π.*

Proof. If $j \leq p_{a-2}$, then the added dots cannot be part of a J_{k-2}, and $q_{a-1} \geq p_{a-2} + 2$; if $p_{a-2} < j \leq p_{a-1}$, then the added dots are the first and last elements of a J_{k-2}, and $q_{a-1} = j + 1$. Thus, regardless of whether $j \leq p_{a-2}$ or $p_{a-2} < j \leq p_{a-1}$, $j' > q_{a-1}$ implies $j' \geq j+2$. If j' is active in σ, consider the involution ρ formed from σ by using this value; if we remove from the graph of ρ the dots that were added to form σ, we obtain the graph of the involution that is obtained by using the value $j' - 2$ in π. Because ρ avoided τ_k so does this involution, so $j' - 2$ is indeed active in π.

Conversely, if j' is not active in σ then—by Lemmas 2.9 and 2.10—using the value j' would produce an involution ρ that contains an instance of τ_k involving the dot in the top row and the dot in the rightmost column of ρ. However, the dot just below the top row lies to the right of the dot in the top row because $j' \geq j + 2$; similarly, the dot in just to the left of the rightmost column lies above the dot in the rightmost column. Thus, these dots cannot participate in the instance of τ_k that we have identified, so deleting them (the dots that were added to π to form σ) produces an involution that again contains τ_k. This involution is the one obtained from π by using the value $j' - 2$, so this was not an active value of π. □

Lemma 2.12. *For $\pi \in \mathcal{I}_n(\tau_k)$ with label $(n, p_1, \ldots, p_a)$, if σ is constructed from π by using the active site $j > p_{a-1}$ and if $(n+2, q_1, \ldots, q_a)$ is the label of σ, then:*

1. *for j' such that $q_{a-1} < j' \leq j + 1$, j' is not active in σ*

2. *for j' such that $j + 2 \leq j' \leq n + 2$, j' is active in σ iff $j' - 2$ is active in π.*

Proof.

1. Let ρ be obtained by using the value j' in σ. Because $j' > q_{a-1}$, the subsquare of the graph of ρ that is defined by these new dots contains a J_{k-4}. Because $j' \leq j+1$, the dots added to π to obtain σ, the dots added to σ, and this J_{k-4} combine to produce a τ_k, so j' is not active in σ.

2. This follows by the same reasoning as in the proof of Lemma 2.11.

□

Lemma 2.13. *For any $\pi \in \mathcal{I}_n(\tau_k)$, $j = n$ is always an active value,* i.e., *we may always add the cycle $(1(n+2))$ to π and obtain another τ_k-avoiding involution.*

Proof. By Lemma 2.9, if we obtain σ by adding $(1(n+2))$ to such a π and if σ contains τ_k, then the value $n+2$ in σ plays the role of k in some instance of τ_k. However, $n+2$ occurs in the first position of σ while k occurs in the second position of τ_k, which is impossible. □

Lemma 2.14. *Given $\pi \in \mathcal{I}_n(\tau_k)$ and $\sigma \in \mathcal{I}_{n+1}(\tau_k)$ obtained from π by adding the fixed point $n+1$:*

1. *$j = 0$ is an active value of σ; and*

2. *for every j, $1 \leq j \leq n+1$, j is an active value of σ iff $j-1$ is an active value of π.*

Proof. We always have 0 as an active value. For $j \geq 1$, let $\rho \in \mathcal{I}_{n+3}$ be the involution obtained from σ by using the value j. The fixed point that was added to create σ (now $(n+2)$ in ρ) cannot be involved in any occurrence of τ_k in ρ. The involution obtained by deleting this fixed point from ρ is the same as the one constructed using the value $j-1$ in π; thus ρ avoids τ_k (and j is active in σ) iff $j-1$ is active in π. □

Lemma 2.15. *All values b, $0 \leq j \leq p_{a-1}$, are active values.*

Proof. If we add the cycle $(m(n+2))$ to $\pi \in \mathcal{I}_n(\tau_k)$ and produce an involution that contains τ_k, then Lemma 2.10 implies that for $j = n+1-m$ the $j \times j$ square in the upper-right corner of π contains J_{k-4}. This in turn implies that $j > p_{a-1}$. □

We may now return to the proof of Thm. 2.6.

Proof of Thm. 2.6. We now describe the multiset of labels of the children of an involution $\pi \in \mathcal{I}_n(\tau_k)$ whose label is $(n, p_1, \ldots, p_a)$. (As in the lemmas above, we make use of the label definitions from Def. 2.7.)

We may always add the fixed point $(n+1)$ to π and obtain an involution in $\mathcal{I}_{n+1}(\tau_k)$ because the added point (as both the largest and rightmost value in the involution) cannot be part of any τ_k. This added point forms a J_1 but cannot be part of any J_i for $i > 1$. Lemma 2.14 implies that this new involution has $p_a + 1$ children in $\mathcal{T}(\tau_k)$, so its label is $(n+1, p_1+1, \ldots, p_a+1)$ if k is even and $(n+1, 0, p_2+1 \ldots, p_a+1)$ if k is odd.

We now compute the labels of the depth-2 children of π. Given $\pi \in \mathcal{I}_n(\tau_k)$ with label $(n, p_1, \ldots, p_a)$, let σ be the depth-2 child of π that is constructed using the active value j of π, and let $(n+2, q_1, \ldots, q_a)$ be the label of σ. The same arguments as in the proof of Theorem 2.3 apply to the computation of $q_1, \ldots, q_{a-1}$, so we only need to compute the total number of depth-2 children of σ. By Lemma 2.15 this equals $q_{a-1}+1$ plus the number of active values of σ that are greater than q_{a-1}.

If $j \leq p_{a-2}$, then $q_{a-1} = p_{a-1}+2$; for every j' such that $q_{a-1} < j' \leq n+2$, Lemma 2.11 implies that j' is active in σ iff $j'-2$ is active in π. As a result, $q_a - q_{a-1} = p_a - p_{a-1}$, so $q_a = p_a + 2$.

If $p_{a-2} < j \leq p_{a-1}$, then the number of active sites in σ that are to the left of the dot at height $n+2$ (*i.e.*, the number of active values of σ that are greater than $q_{a-1} = j+1$) equals $q_a - (q_{a-1}+1)$; by Lemma 2.11 the number in question also equals the total number of active values of π that are greater than or equal to j. This is $p_a - j$, so $q_a - (q_{a-1}+1) = p_a - j$ and $q_a = p_a + 2$.

If $j > p_{a-1}$, then $q_{a-1} = p_{a-1}+1$. By Lemma 2.12, the number of active values j' in σ that are greater than q_{a-1} equals the number of active values of π that are at least j and at most n. When $j = n$ (always an active value by Lemma 2.13) there is one such active value so that $q_a = q_{a-1}+2$. Decreasing j to be the next largest active value in π greater than p_{a-1} produces a child σ of π with one more active value; this process may be continued until j is the smallest active value of π that is greater than p_{a-1}, at which point $q_a = (q_{a-1}+1) + (p_a - (p_{a-1}+1)) = p_a + 1$.

Finally, observe that relabeling the nodes in $\mathcal{T}(\tau_k)$ using the map $(n, p_1, \ldots, p_{a-1}, p_a) \mapsto (n, p_1, \ldots, p_{a-1}, p_a - 1)$ gives labels and rewriting rules as in Equation (1), so $\mathcal{T}(\tau_k)$ is isomorphic to the k-tree as claimed. □

Corollary 2.16. *For $k \geq 5$, the generating trees for involutions avoiding $k(k-1)\ldots 21$ and for involutions avoiding $(k-1)k(k-2)\ldots 312$ are isomorphic.*

Corollary 2.17. *For $k \geq 5$, $k(k-1)\ldots 21 \sim_{\mathcal{I}_n} (k-1)k(k-2)\ldots 312$.*

This isomorphism of generating trees induces a bijection between the involutions avoiding J_k and those avoiding $(k-1)k(k-2)\ldots 312$. It remains open to give a concise description of this bijection, *i.e.*, one that does not require tracing the construction of the involutions.

Figure 2 updates Figure 1 to show, in addition to the labels of nodes as in Definition 2.2 (the first line at each node), the corresponding involutions from $\mathcal{T}(J_5)$ (the second line at each node) and $\mathcal{T}(\tau_5)$ (the third line at each node). Observe that in general, an involution that appears in both $\mathcal{T}(J_5)$ and $\mathcal{T}(\tau_5)$ need not appear in the same place or with the same label.

3 Classification results and open problems

One motivating question for the work in Sec. 2 was whether $45312 \sim_{\mathcal{I}_n} 12345$ as conjectured in [Jag03]. Corollary 2.17 affirmatively answers this conjecture, completing the classification of $\mathcal{S}_5$ into $\sim_{\mathcal{I}_n}$-classes. Table 1 shows the involutions in $\mathcal{S}_5$; the sets of patterns are symmetry classes, while symmetry classes that are not separated by horizontal lines are in the same $\sim_{\mathcal{I}_n}$-class. (As noted in [Jag03], the only $\mathcal{I}_n$-Wilf-equivalences involving non-involutions in $\mathcal{S}_5$ are $12453 \sim_{\mathcal{I}_n} 21453$ and those implied by symmetry.)

Our work also affirmatively answers two questions posed in [Jag03], namely whether $123456 \sim_{\mathcal{I}_n} 564312$ and whether there is a more general result that implies both $45312 \sim_{\mathcal{I}_n} 12345$ and $123456 \sim_{\mathcal{I}_n} 564312$. This work does not answer the question from [Jag03] of whether $123456 \sim_{\mathcal{I}_n} 456123$, the one unresolved $\mathcal{I}_n$-Wilf-equivalence between 123456 and

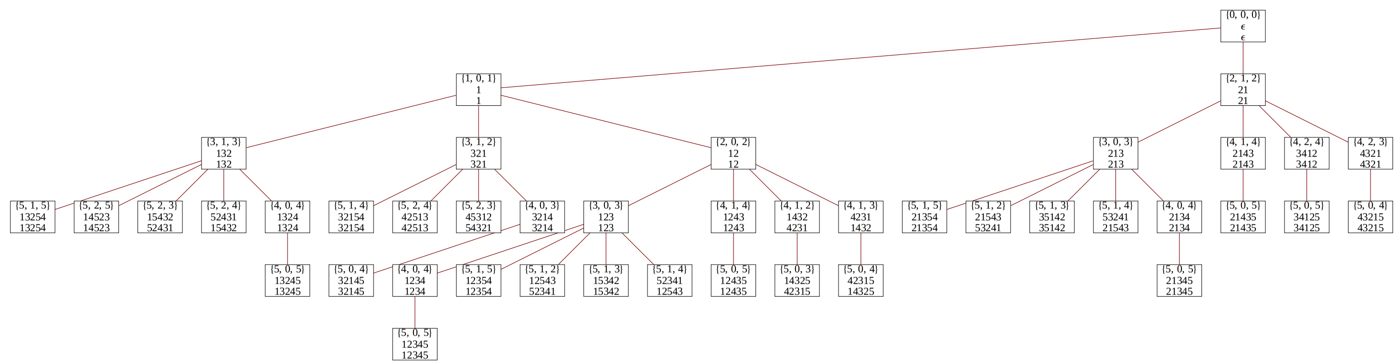

Figure 2: Levels 0–5 of the 5-tree with node labels and the corresponding permutations in $\mathcal{T}(J_5)$ and $\mathcal{T}(\tau_5)$.

τ	$I_6(\tau)$	$I_7(\tau)$	$I_8(\tau)$	$I_9(\tau)$	$I_{10}(\tau)$	$I_{11}(\tau)$
$\{35142, 42513\}$	70	195	582	1725	5355	16510
$\{14325\}$	70	196	587	1757	5504	17220
$\{12435, 13245\}$ $\{13254, 21435\}$	70	196	587	1759	5512	17290
$\{12345\}$ $\{54321\}$ $\{12354, 21345\}$ $\{12543, 32145\}$ $\{21354\}$ $\{21543, 32154\}$ $\{15432, 43215\}$ $\{45312\}$	70	196	588	1764	5544	17424
$\{52431, 53241\}$	70	196	588	1764	5544	17426
$\{52341\}$	70	196	589	1773	5604	17768
$\{14523, 34125\}$	70	197	592	1791	5644	17900
$\{15342, 42315\}$	70	197	593	1797	5685	18101

Table 1: Classification of involutions in $\mathcal{S}_5$ into $\sim_{\mathcal{I}_n}$-classes.

other patterns in $\mathcal{S}_6$. (This question has been affirmatively answered by Dukes, Jelínek, Mansour, and Reifegerste [DJMR09] in their recent work noted above.)

The patterns J_k and τ_k are already known to be Wilf-equivalent for every k (*i.e.*, for pattern avoidance by permutations in general). The results here, that these patterns are also $\mathcal{I}_n$-Wilf-equivalent, provides another example of parallels between $\mathcal{S}_n$ and $\mathcal{I}_n$; the study of ways in which these (and other pairs of) classes of permutations resemble each other remains an interesting general question.

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