A Note on Polyhedral Relaxations for the Maximum Cut Problem

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Abstract

We consider three well-studied polyhedral relaxations for the maximum cut problem: the metric polytope of the complete graph, the metric polytope of a general graph, and the relaxation of the bipartite subgraph polytope. The metric polytope of the complete graph can be described with a polynomial number of inequalities, while the latter two may require exponentially many constraints. We give an alternate proof of a theorem of Barahona that states that the metric polytope of a general graph is a projection of the metric polytope of the complete graph. We then give an alternate proof of a theorem of Poljak that states that for any non-negative cost function, the optimal objective value over the relaxation of the bipartite subgraph polytope equals the optimal objective value over the metric polytope. Both proofs are based on the same technique: the separation oracle for the metric polytope of a general graph due to Barahona and Mahjoub. These proofs yield a simple, combinatorial method for proving that three well-studied polyhedral upper bounds on the value of the maximum cut are the same for graphs with non-negative edge weights.

1 Introduction

Given an undirected, weighted graph, the *maximum cut* problem is to find a bipartition of the vertices that maximizes the weight of the edges crossing the partition, i.e. a maximum bipartite subgraph. We consider three well-known polyhedral relaxations for the maximum cut problem. Theorems due to Barahona [Bar93] and Poljak [Pol92] imply that optimization over these three polytopes gives the same upper bound on the value of a maximum cut for graphs with non-negative edge weights. In this paper, we give new proofs for both of these theorems that are based on the same technique: the separation algorithm for the metric polytope. These proofs are similar to the proofs that two well-studied linear programming relaxations for the maximum acyclic subgraph problem provide the same upper bound for graphs with non-negative edge weights [NV01].

2 The Metric Polytope of a General Graph

The first polytope we address is the *metric polytope of a general graph*. This polytope was introduced by Barahona and Mahjoub [Bar83, BM86]. Following the notation that Poljak and Tuza

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use in their survey on the maximum cut problem [PT95], we refer to the polytope defined by the following inequalities for a specified graph $G = (V, E)$ as $P^{met}(G)$.

$$\begin{aligned} \sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e &\leq |F| - 1 \quad \forall \textit{ cycles } C, F \subseteq C, |F| \textit{ odd} \qquad (1) \\ 0 \leq x_e &\leq 1 \quad \forall e \in E. \end{aligned}$$

Constraint (1) can be re-written as:

$$\sum_{e \in F}(1 - x_e) + \sum_{e \in C \setminus F} x_e \geq 1 \quad \forall \textit{ cycles } C, F \subseteq C, |F| \textit{ odd.} \qquad (2)$$

Barahona and Mahjoub showed that there is a polynomial-time separation oracle for this polytope [BM86]. Given a graph $G = (V, E)$ and a point $\mathbf{x} = \{x_e\}$, they gave the following efficient algorithm for checking if $\mathbf{x}$ belongs to $P^{met}(G)$.

First, create a new weighted bipartite graph $B(G)$. The vertex set of $B(G)$ is $V \cup V'$: for each vertex $i \in V$, we have $i \in V$ and $i' \in V'$. For each edge $e = (i, j) \in E$, add the edges (i, j) and (i', j'), each with weight x_e, and edges (i, j') and (j, i'), each with weight $1 - x_e$. This construction is illustrated in Figure 1.

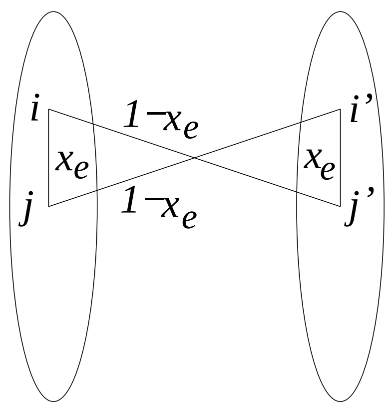

Figure 1: The edges $(i, j), (i', j'), (i, j')$, and (i', j) in $B(G)$ that correspond to the edge (i, j) in G.

Now we can check to see if the point $\mathbf{x}$ satisfies constraint (2) by finding the shortest path from i to i' for each vertex $i \in V$. Note that the path from i to i' in $B(G)$ corresponds to a cycle in G and this path contains an odd number of edges that are included with edge weight $1 - x_e$. Thus, for each cycle C in G containing vertex i, if there is a subset $F \subseteq C$ of the edges such that $|F|$ is odd and constraint (2) is violated, then the shortest path from i to i' will be strictly less than one.

3 The Metric Polytope of a Complete Graph

The *metric polytope of a complete graph* has a variable for every pair of vertices $i, j \in V$. This polytope has constraints for each cycle C in the complete graph such that $|C| = 3$ and for each subset $F \subseteq C$, where $|F| = 1, 3$. Following the notation of Poljak and Tuza [PT95], we refer to the polytope defined by the following inequalities as P_n^{met}, where n indicates the number of vertices in a specified graph G.

$$\begin{aligned}
x_{ij} + x_{jh} + x_{hi} &\leq 2 \quad \forall h, i, j \in V && (3)\\
x_{ij} + x_{jh} - x_{hi} &\geq 0 && (4)\\
x_{ij} - x_{jh} + x_{hi} &\geq 0 && (5)\\
-x_{ij} + x_{jh} + x_{hi} &\geq 0 && (6)\\
x_{ij} &\geq 0 \quad \forall i, j \in V.
\end{aligned}$$

Given a point $\mathbf{x} = \{x_{ij}\}$, it is clear that we can efficiently check if x belongs to P_n^{met} since we need only check a polynomial number of constraints.

Theorem 1 [Barahona [Bar93], [PT95]]
Let $G = (V, E)$ be a graph on n vertices. Then $P^{met}(G)$ is a projection of P_n^{met} to the subspace $R^{|E|} \subseteq R^{\binom{n}{2}}$.

For one direction of our proof of Theorem 1, we will use Remark 6.2 from [Bar93]. We restate this remark and its proof here for the sake of completeness.

Lemma 1 [Barahona [Bar93], Remark 6.2]
If a cycle C has a chord e, then any inequality of P^{met} associated with C is a sum of two other inequalities associated with the two new cycles obtained by adding edge e to C. This proves that the inequalities associated with C are redundant.

Proof: Consider

$$x(F) - x(C \setminus F) \leq |F| - 1, \quad F \subseteq C, \quad |F| \text{ odd.} \tag{7}$$

Let C_1 and C_2 be the new cycles obtained by adding the chord e to C. Let $F_i = C_i \cap F, i = 1, 2$. Suppose that $|F_1|$ is odd, then constraint (7) is the sum of the next two constraints:

$$x(F_1) - x(C_1 \setminus F_1) \leq |F_1| - 1,$$
$$x_e + x(F_2) - x(C_2 \setminus [F_2 \cup \{e\}]) \leq |F_2|.$$

□

Proof of Theorem 1: If a point $\mathbf{x} = \{x_{ij} \mid i, j \in V\}$ belongs to P_n^{met}, then $\mathbf{y} = \{x_{ij} \mid ij \in E\}$ belongs to $P^{met}(G)$. This follows from Lemma 1.

Now we show that if a point $\mathbf{y} = \{x_{ij} \mid ij \in E\}$ belongs to $P^{met}(G)$, then there exists a set of values $\{x_{ij} \mid ij \notin E\}$ such that $\mathbf{x} = \{x_{ij} \mid i,j \in V\}$ belongs to P_n^{met}. Consider an edge $(h,k) \notin E$. We will determine a value for x_{hk} such that the point $(\mathbf{y}, x_{hk})$ belongs to $P^{met}(G + \{(h,k)\})$. By successively adding edges, we will find a point $\mathbf{x} = (\mathbf{y}, \{x_{ij} \mid ij \notin E\})$ such that $\mathbf{x}$ belongs to $P^{met}(K_n)$ implying that $\mathbf{x}$ belongs to P_n^{met}.

To find the value for x_{hk}, we construct a bipartite graph $B(G)$ using the method shown in Section 2. Find the shortest path from h to k in $B(G)$. Denote this value as α and assign $x_{hk} = \min\{1, \alpha\}$. We can show that in the graph with edge set $E+\{(h,k)\}$, no constraints (2) from $P^{met}(G+\{(h,k)\})$ are violated. Suppose there is a violated constraint. By induction all paths in $B(G+\{(h,k\})$ from i to i' that do not contain an edge from the set $\{(h,k),(h',k'),(h',k),(h,k')\}$ do not correspond to violated constraints. So the path associated with the violated constraint must contain an edge from this set.

Without loss of generality, we can assume the violated constraint corresponds to a path from h to h'. This is because, we can assume the violated constraint corresponds to a path from some i to i'. (Recall that a violated constraint corresponds to a path from i to i' in $B(G)$ with cost less than 1.) Then it must contain a path from i to h and from h to i'. But there is a path in $B(G)$ from i' to h' that has the same cost as the path from i to h. Thus, we can consider the path from h to i' and from i' to h' as the path corresponding to the violated constraint. We will refer to this path as p.

Suppose the path p contains edge (h,k') or edge (h',k). Without loss of generality, we assume it contains edge (h,k'). Then it also contains the shortest path from k' to h', which is equal to the shortest path from h to k. This path has the value $\min\{1,\alpha\}$. Since edge (h,k') has value $1 - \min\{1,\alpha\}$, the value of path p is at least one.

Suppose the path p contains edge (h,k) or edge (h',k'). Each of these edges could be replaced by the edges that comprise the shortest path from h to k or from h' to k' in $B(G)$. If x_{hk} had value less than one, then the resulting path from h to h' in $B(G)$ either has cost equal to the cost of p, which by assumption is less than one. Thus, a path in $B(G)$ corresponds to a violated constraint in $P^{met}(G)$, which contradicts the inductive hypothesis. So the point $(\mathbf{y}, x_{hk})$ does not violate any constraint in $P^{met}(G + \{(h,k)\})$. □

4 The Bipartite Subgraph Polytope

The *bipartite subgraph polytope* is another polytope associated with the maximum cut problem that was introduced by Barahona, Grötschel and Mahjoub [BGM85]. For a given graph G, this polytope contains a constraint for every odd-cycle. Following the notation of Poljak and Tuza [PT95], we refer to the relaxation of the bipartite subgraph polytope as $Q(G)$:

$$\begin{aligned} \sum_{e \in C} x_e \quad &\leq \quad |C| - 1 \quad \forall \textit{ odd cycles } C \qquad (8) \\ 0 \quad \leq \quad x_e \quad &\leq \quad 1 \quad \forall ij \in E. \end{aligned}$$

For a non-negative cost function c, we use the following definitions.

Define ω_1 to be the maximum value of $c \cdot \mathbf{x}$ over $\mathbf{x} \in P_n^{met}$.

Define ω_2 to be the maximum value of $c \cdot \mathbf{x}$ over $\mathbf{x} \in P^{met}(G)$.

Define ω_3 to be the maximum value of $c \cdot \mathbf{x}$ over $\mathbf{x} \in Q(G)$.

Theorem 2 [Poljak [Pol92], Theorem 4]
For a graph $G = (V, E)$ with non-negative edge weights, the upper bounds on the value of the maximum cut of G provided by ω_1, ω_2 and ω_3 are all equal.

Proof: By Theorem 1, ω_1 and ω_2 are equal. We will prove that ω_2 and ω_3 are equal, which proves the theorem. Consider a point $\mathbf{x}$ in $P^{met}(G)$ that has an objective value $c \cdot \mathbf{x}$. It is also a feasible point in $Q(G)$ since the constraints in $Q(G)$ are a subset of the constraints that define $P^{met}(G)$. Thus, there is a point in $Q(G)$ with the same objective value.

Consider a point $\mathbf{x}$ in $Q(G)$ that has optimal value c^* for some non-negative cost function. For each edge $e \in E$ such that $c_e = 0$, we increase the value of x_e so that it is as large as possible and does not violate any of the odd-cycle constraints. We can greedily increase the value of every such edge x_e until these values are maximal. Note that the resulting point $\mathbf{x}$ also belongs to $Q(G)$ and still has objective value c^*.

Suppose that there is some constraint in $P^{met}(G)$ that is violated by the solution $\mathbf{x}$. Then for some vertex $i \in V$, there is some path in $B(G)$ from i to i' such that the value of the path in $B(G)$ is strictly less than one. Consider such a path. We say an edge in $B(G)$ is a *cross edge* if it goes from V to V'. If all the edges in the path corresponding to the violated constraint are cross edges in $B(G)$, then they must have value at least one in G, since a path in $B(G)$ comprised entirely of cross edges corresponds to the sum $\sum_{e \in C}(1 - x_e)$ where C is an odd cycle in G.

So it must be the case that some edge in the path p from i to i' in $B(G)$ is not a cross edge. We will denote this edge as $f = (h, k)$. Since the solution $\mathbf{x}$ is optimal for $Q(G)$, there must be some odd cycle C_f containing edge f that is tight, i.e. has value $|C_f| - 1$ (otherwise we could have increased x_f). Thus, the sum of the complementary values of edges in C_f must be exactly one, i.e. $\sum_{e \in C_f}(1 - x_e) = 1$. This implies that $\sum_{e \in C_f \setminus \{f\}}(1 - x_e) = x_f$. So we replace edge f in the path p with the (even-length) path from h to k of the cross edges. Note that this preserves the value of the shortest path from i to i' in $B(G)$. We do this for all non-cross edges in the path p. When we are done, we have a path from i to i' in $B(G)$ containing only cross edges that has the same value as the shortest path from i to i' in $B(G)$. Since this path corresponds to an odd cycle in G, these cross edges or complimentary edges must sum to at least one. □

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