“Robust” Lower Bounds

for Communication and Stream Computation

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Graham Cormode AT&T Labs
Andrew McGregor UC San Diego

Communication Complexity

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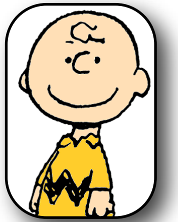
- *Goal:* Evaluate $f(x_1, \dots, x_n)$ when input is split among p players:

$x_1 \dots x_{10}$

$x_{11} \dots x_{20}$



$x_{21} \dots x_{30}$

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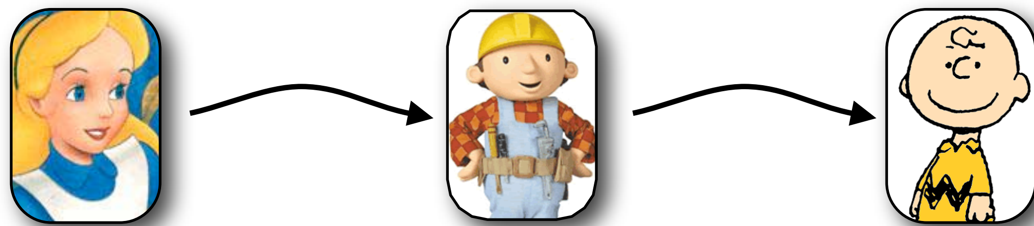

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Is f hard for many splits or only hard for a few bad splits?

Previous work on worst and best partitions.

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- *Consider random partitions:*

Define error probability over coin flips and random split.

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- ***Goal:*** Evaluate $f(x_1, \ldots, x_n)$ given sequential access:

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  Upper Bounds: e.g., stream of i.i.d. samples.

  Lower Bounds: is a "hard" problem hard in practice?

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- Random-partition-CC bounds give random-order bounds

Results

- *t-party Set-Disjointess:* Any protocol for $\Omega(t^2)$-player random-partition requires $\Omega(n/t)$ bits communication.

  ∴ 2-approx. for k^{th} freq. moments requires $\Omega(n^{1-3/k})$ space.

- *Median:* Any p-round protocol for p-player random-partition requires $\Omega(m^{f(p)})$ where $f(p)=1/3^p$

  ∴ Polylog(m)-space algorithm requires $\Omega(\log \log m)$ passes.

- *Gap-Hamming:* Any one-way protocol for 2-player random-partition requires $\Omega(n)$ bits communicated.

  ∴ $(1+\varepsilon)$-approx. for F_0 or entropy requires $\Omega(\varepsilon^{-2})$ space.

- *Index:* Any one-way protocol for 2-player random-partition (with duplicates) requires $\Omega(n)$ bits communicated.

  ∴ Connectivity of a graph $G=(V, E)$ requires $\Omega(|V|)$ space.

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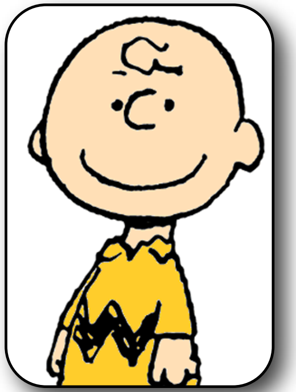

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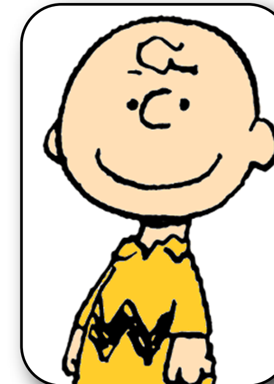

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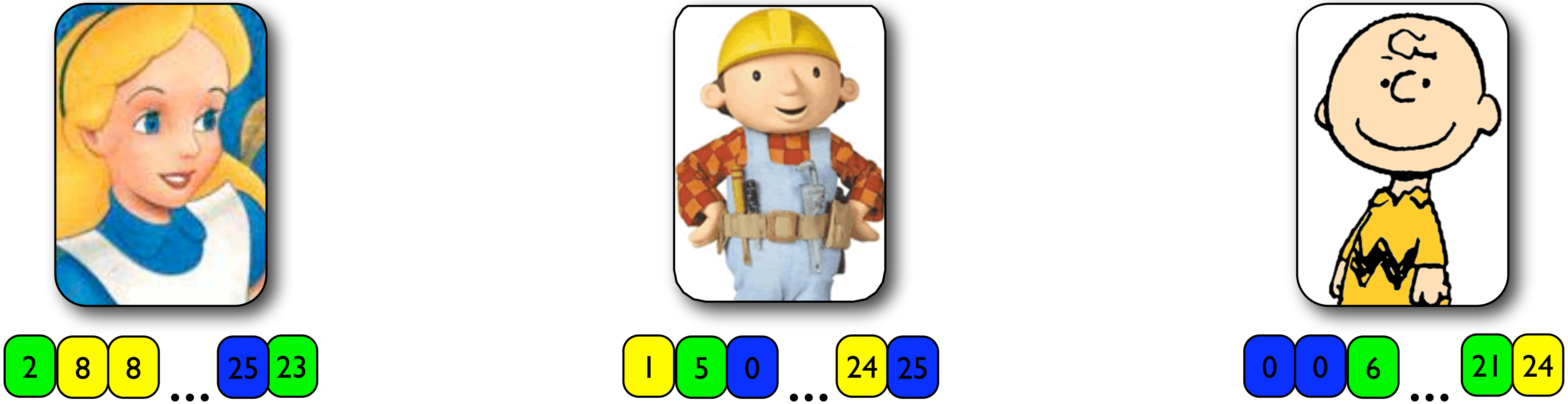

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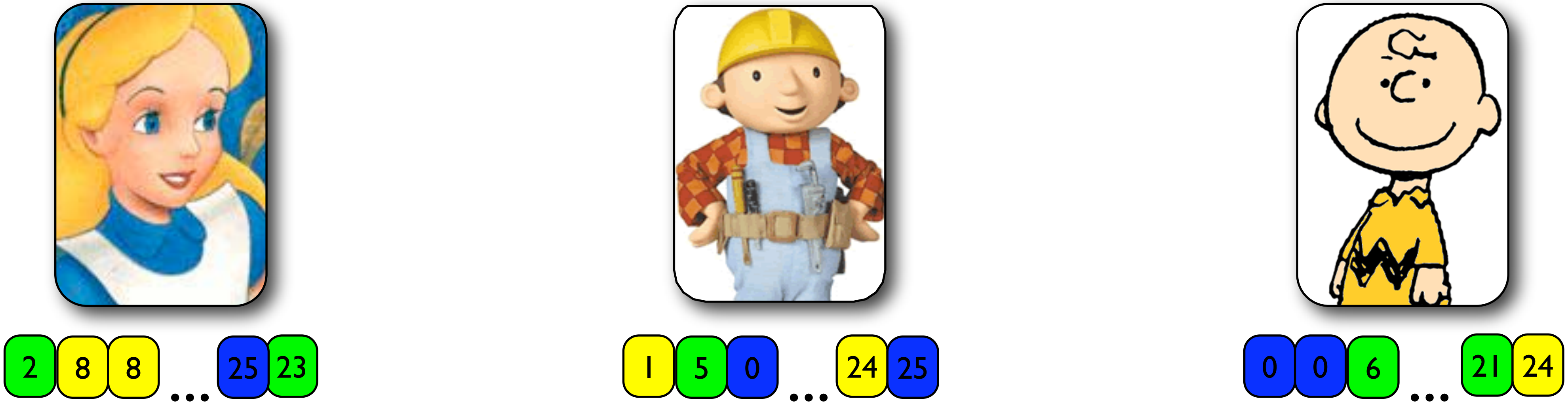

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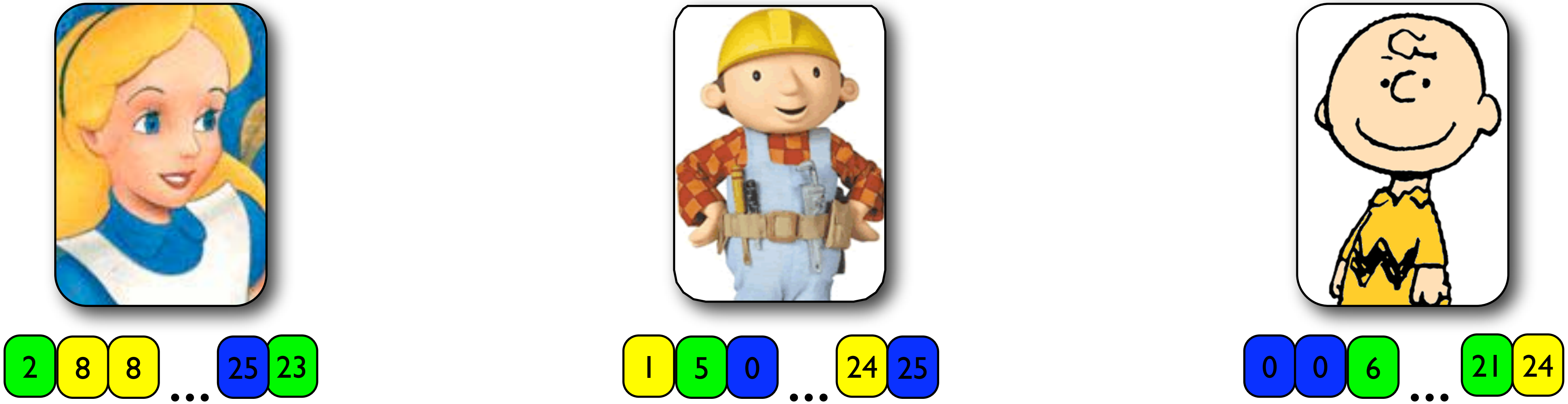

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- *Consider random input and public coins:*

  Issue #1: Need independence of input and partition.

  Issue #2: Generalize information statistics techniques.

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Multi-Party Set-Disjointness

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- *Thm:* $\Omega(n/t)$ bound for random partition for $\Omega(t^2)$ players.

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- *Necessary Generalization:*

  Step 1: Condition "icost" on public coins.

  Step 2: Error of Π' is best δ+Birthday(t,p) error protocol.

  Step 3: Generalize result for public-coin protocols.

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- <u>*Open Problem:*</u> $\Omega(n^{1-2/k})$ bound for random order?

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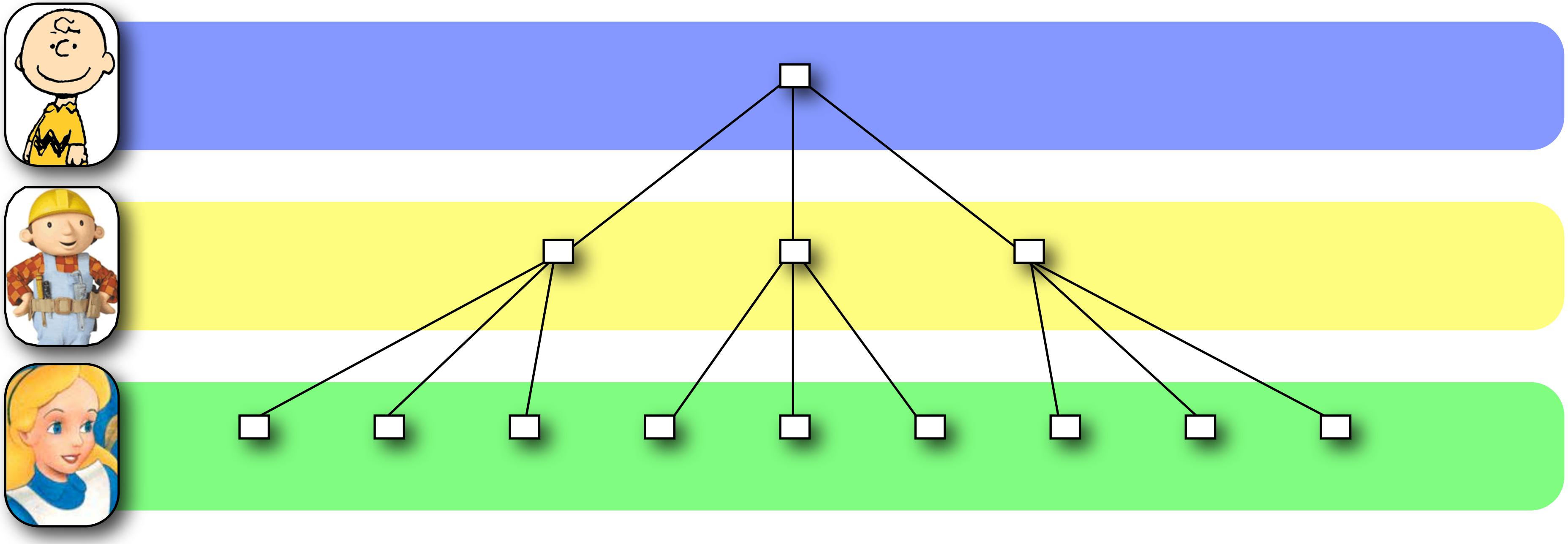
- *Our result:* Using random-partition-CC techniques we get simpler and tighter pass/space trade-offs...

Tree Pointer Jumping (TPJ)...



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  if v is an internal node: f maps v to a child of v
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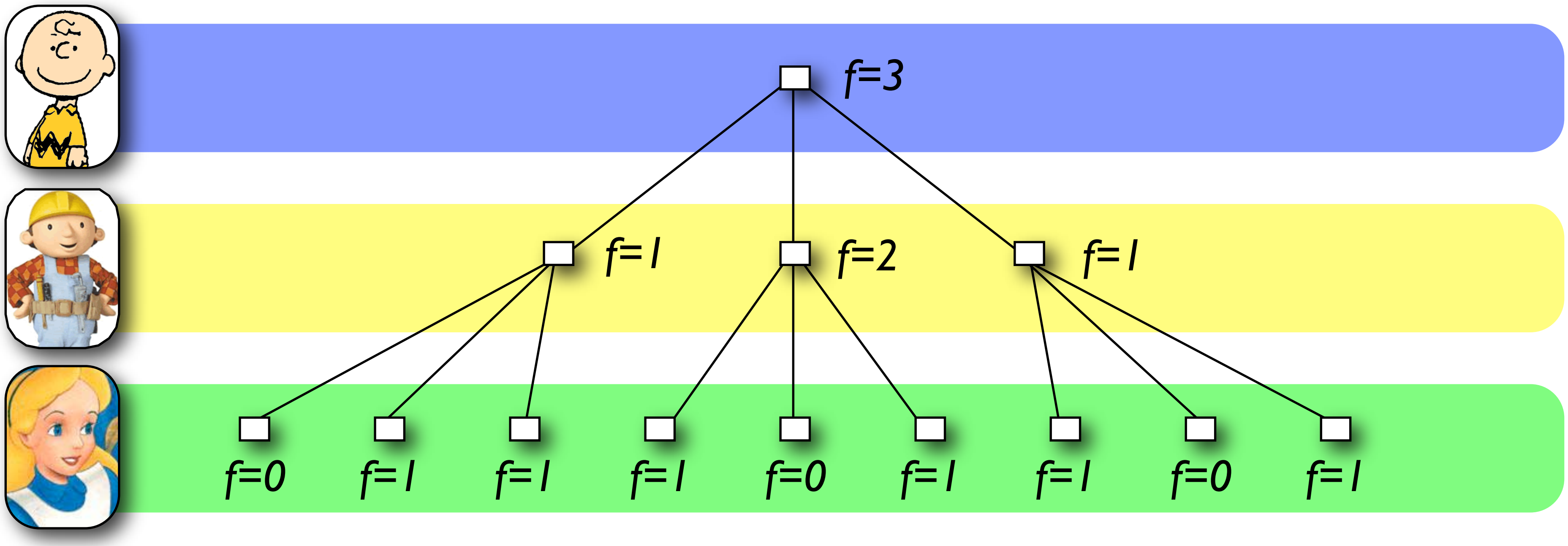

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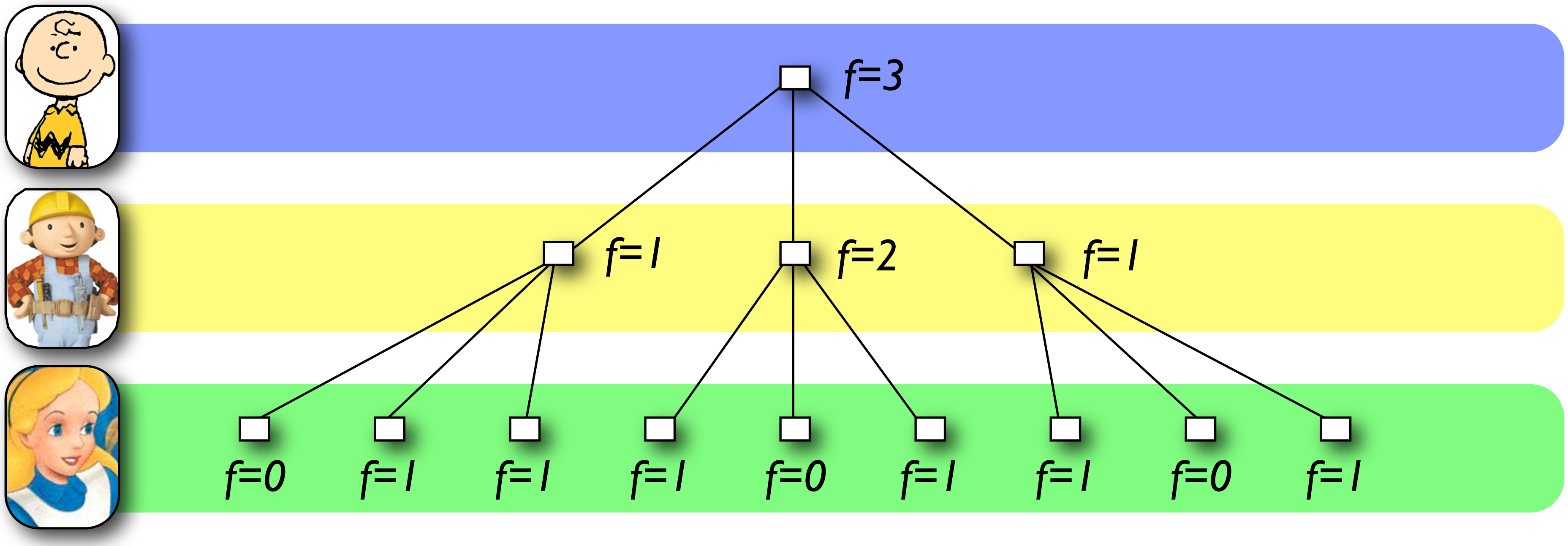

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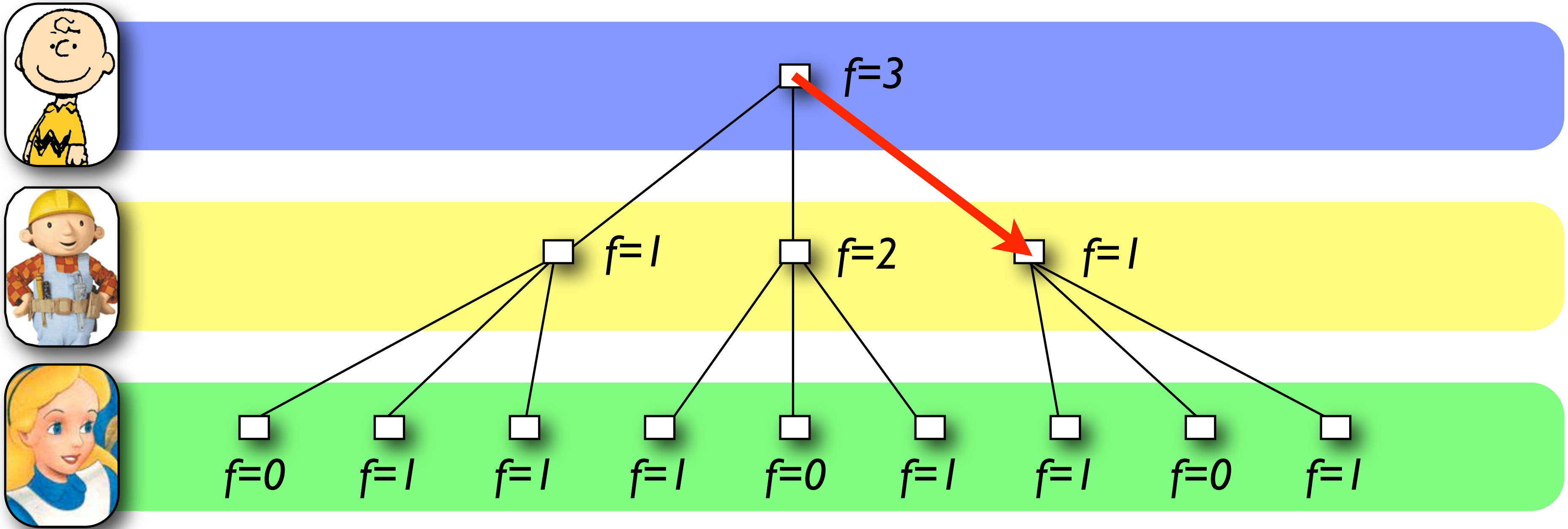

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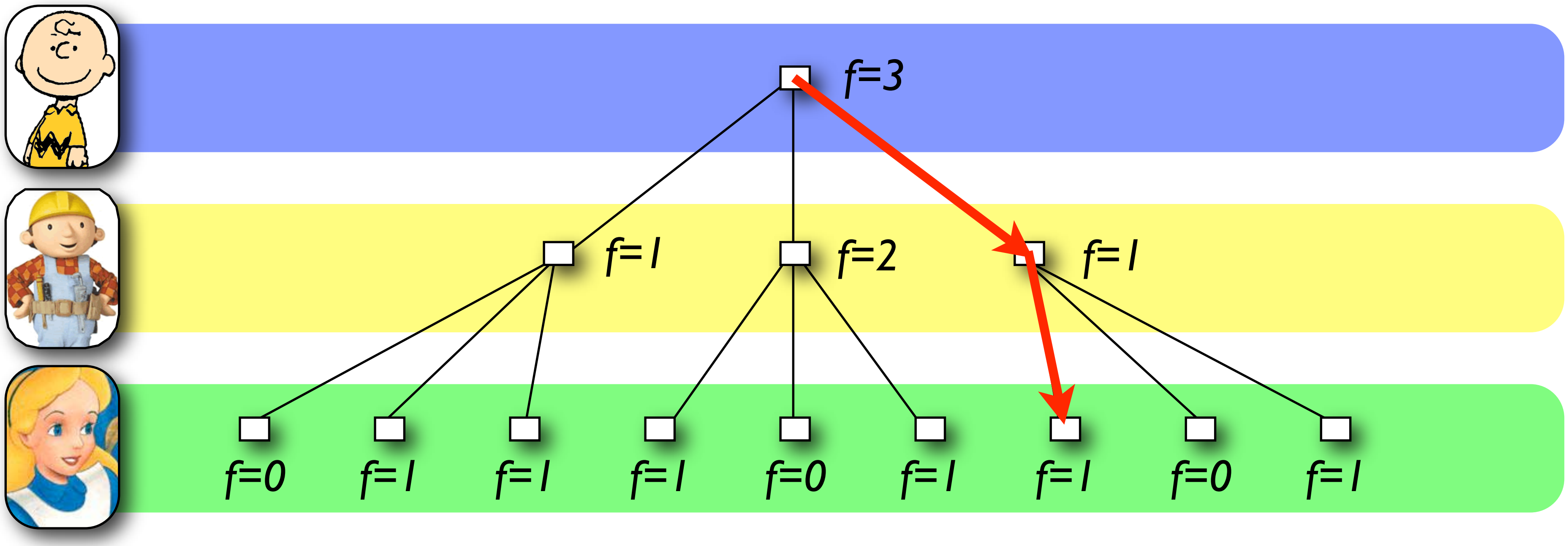

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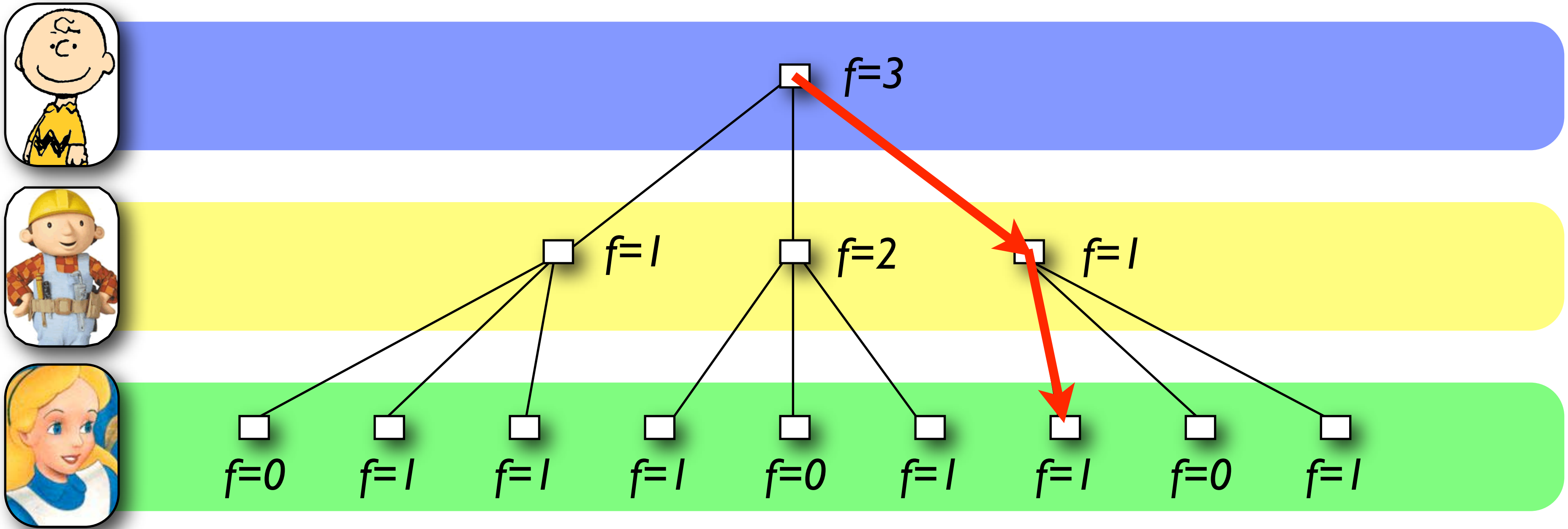

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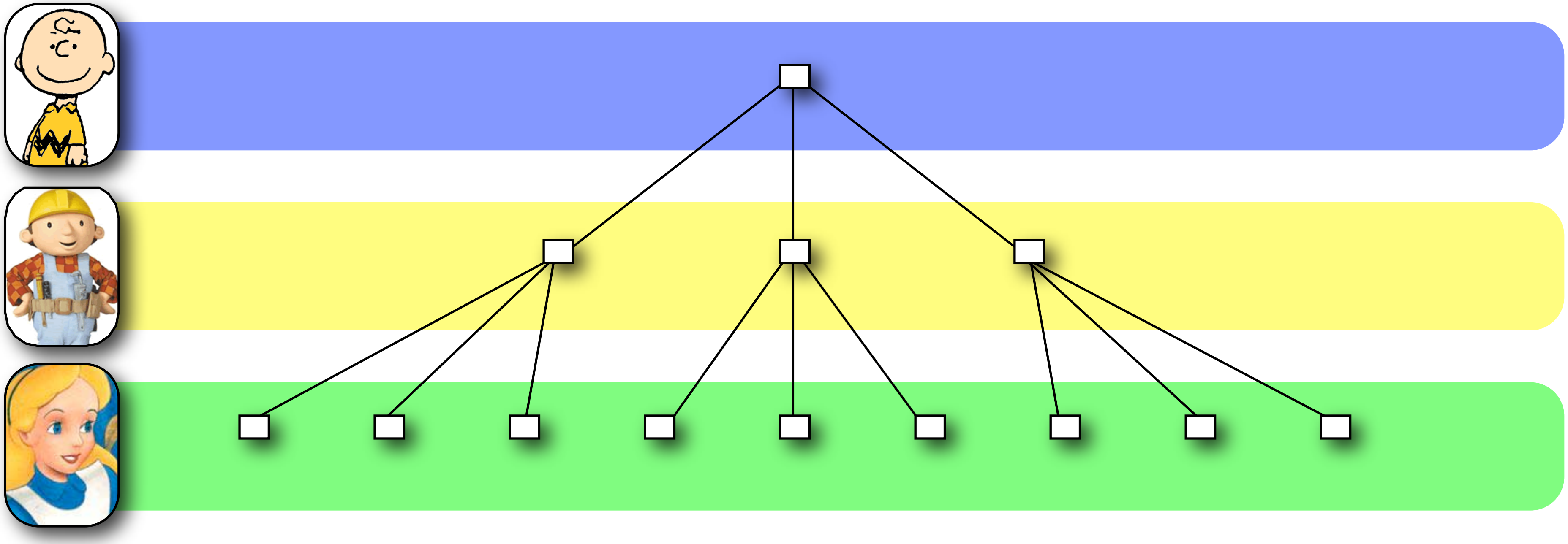

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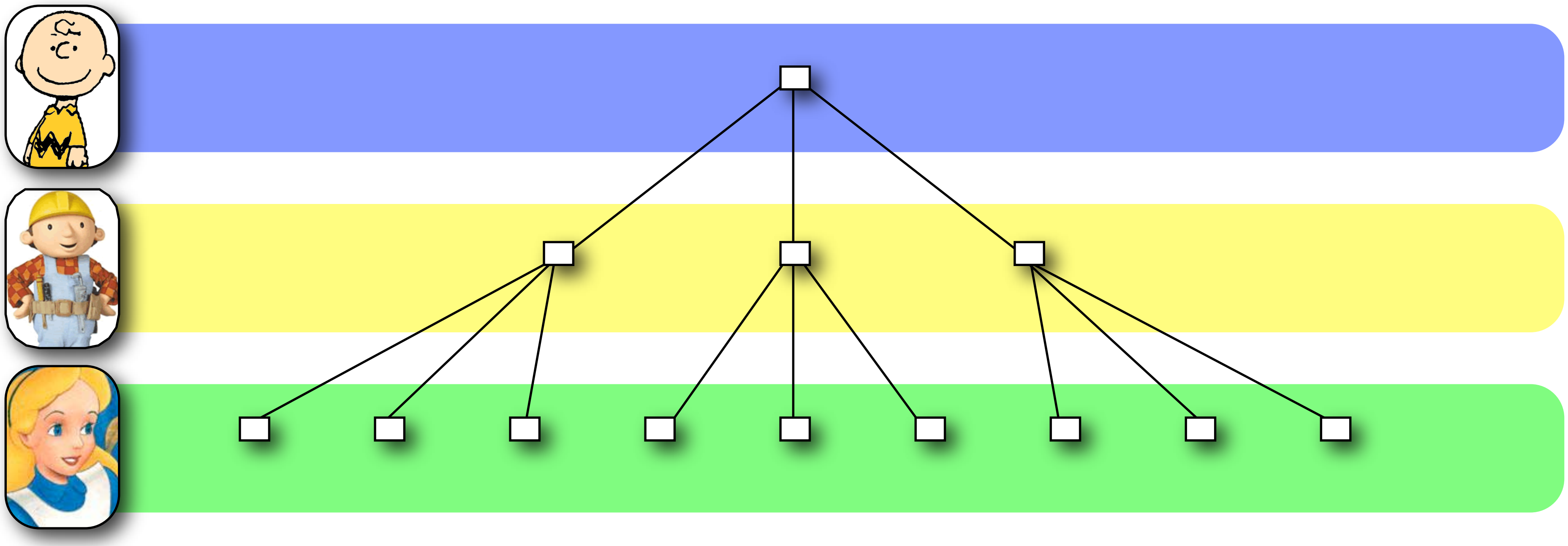

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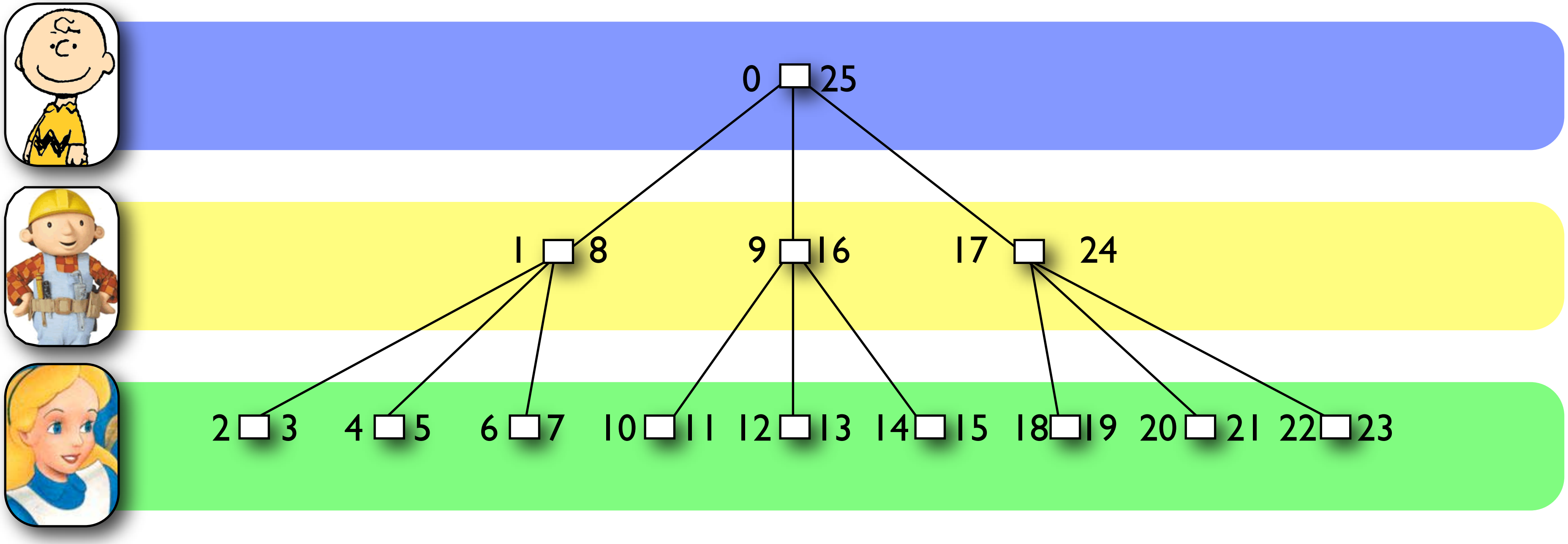

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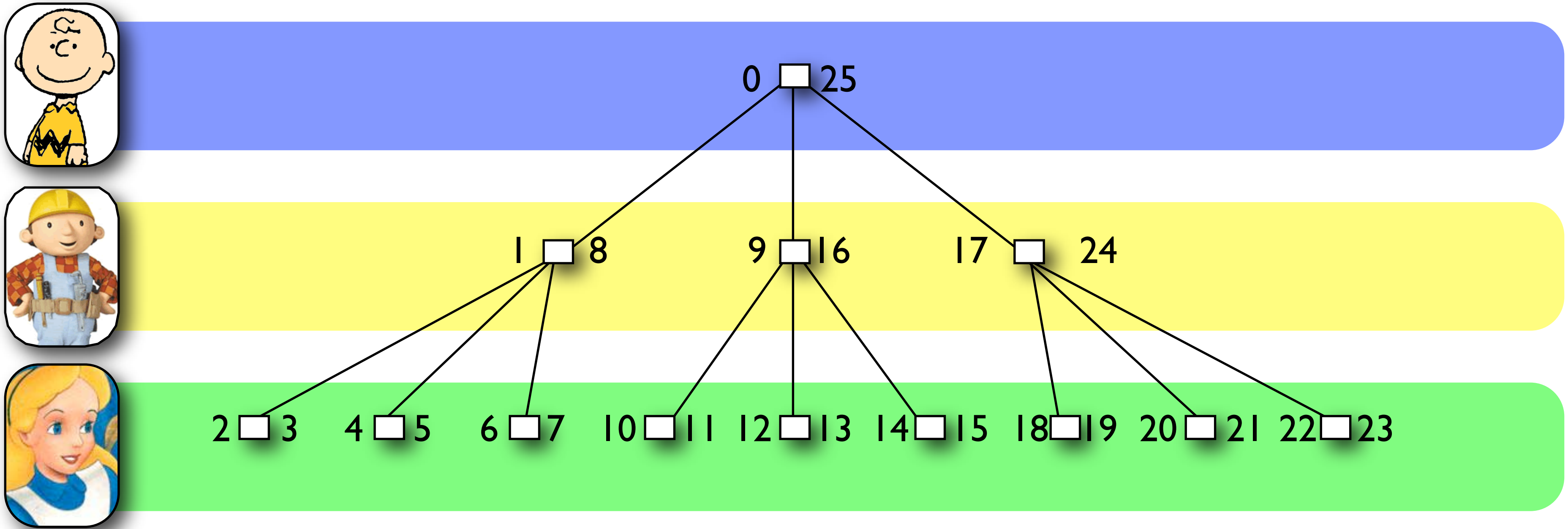

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- *For each node:* Generate multiple copies of $\alpha(v)$ and $\beta(v)$ such that median of values corresponds to TPJ solution.

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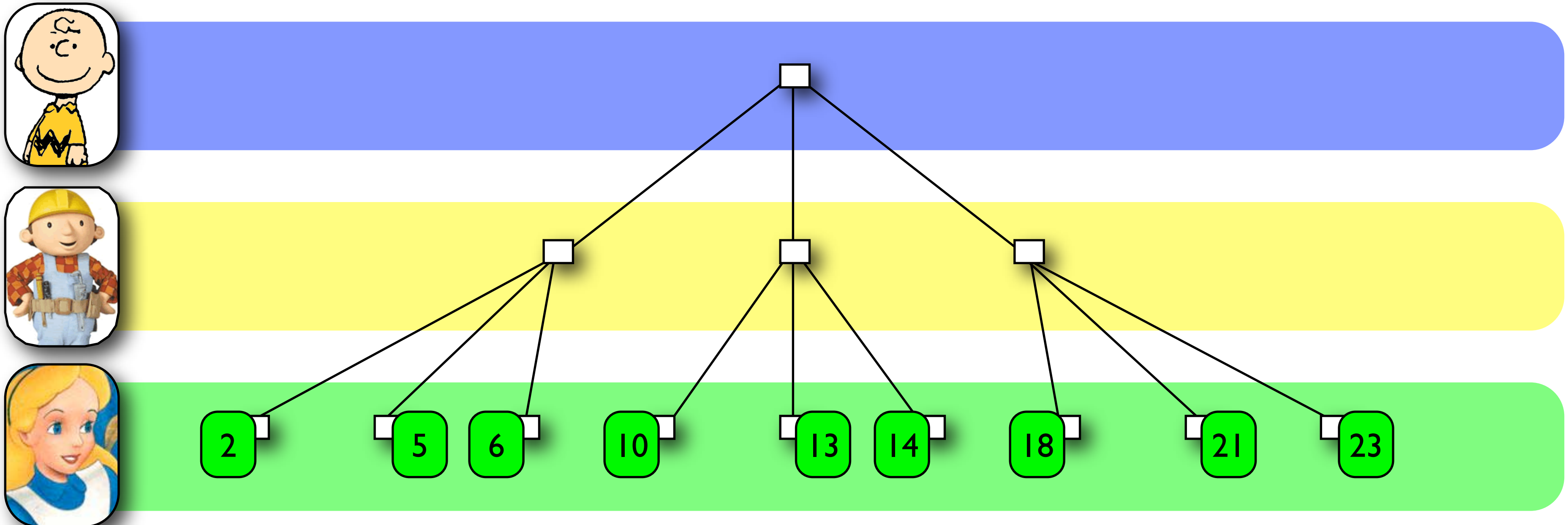

- With each node v associate two values $\alpha(v) < \beta(v)$ such that $\alpha(v) < \alpha(u) < \beta(u) < \beta(v)$ for any descendent u of v.
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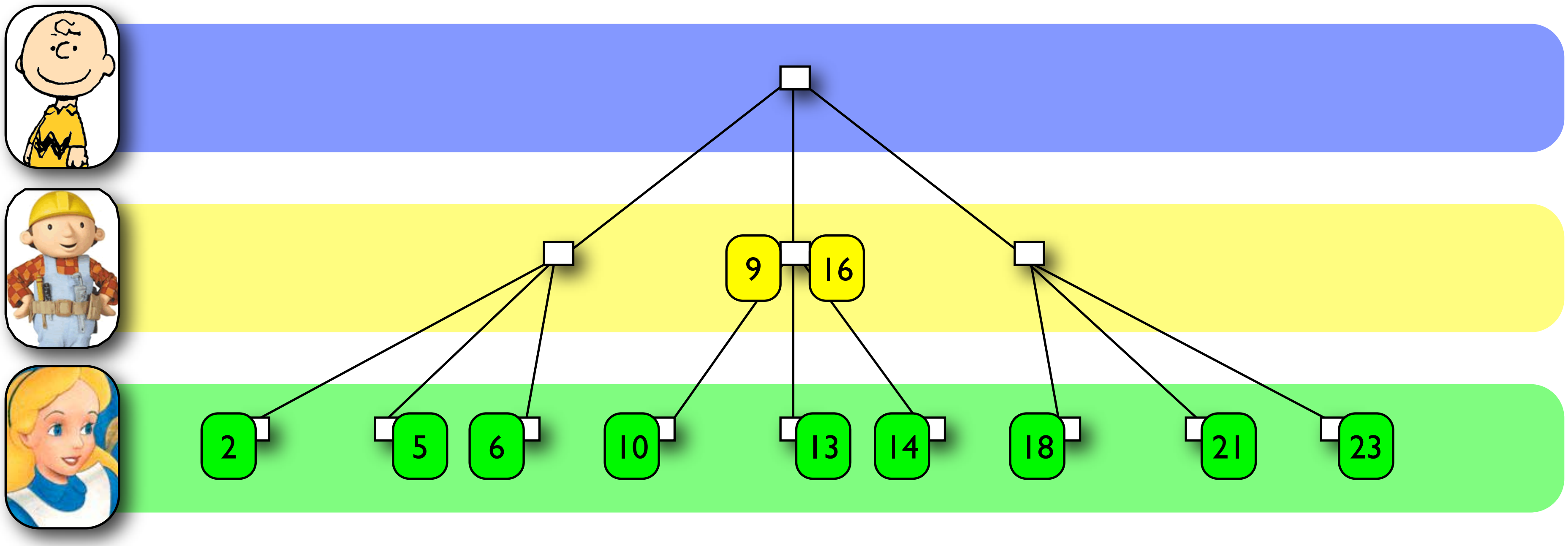

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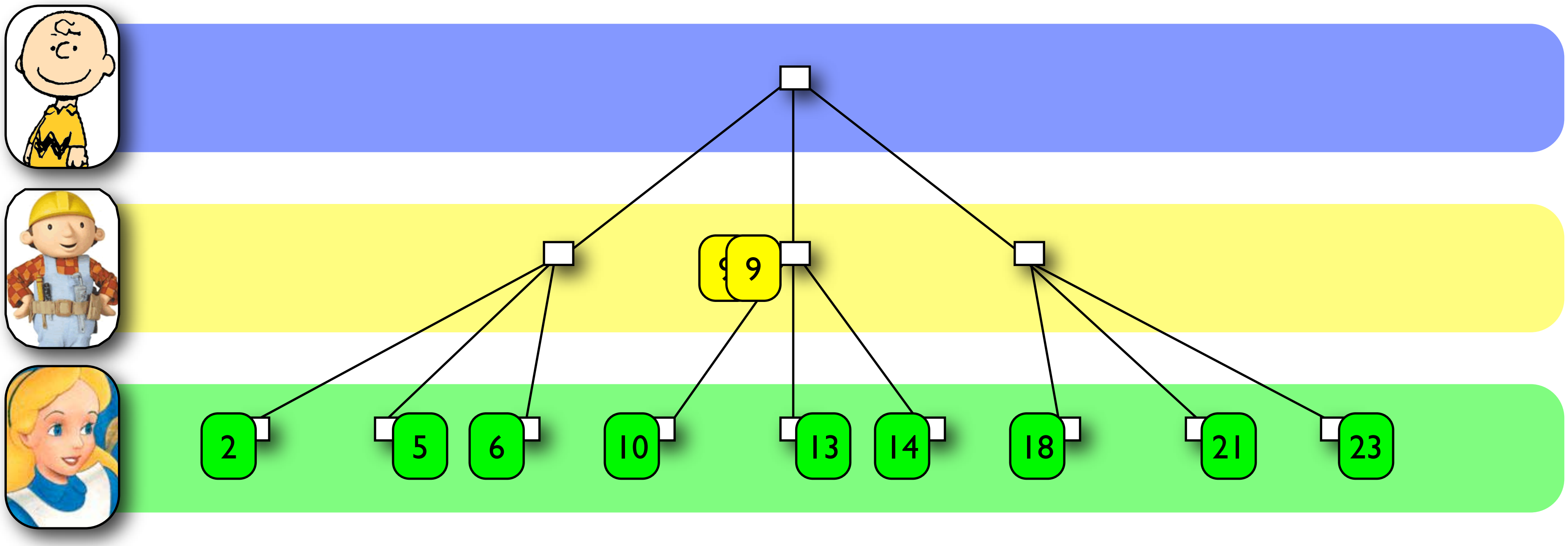

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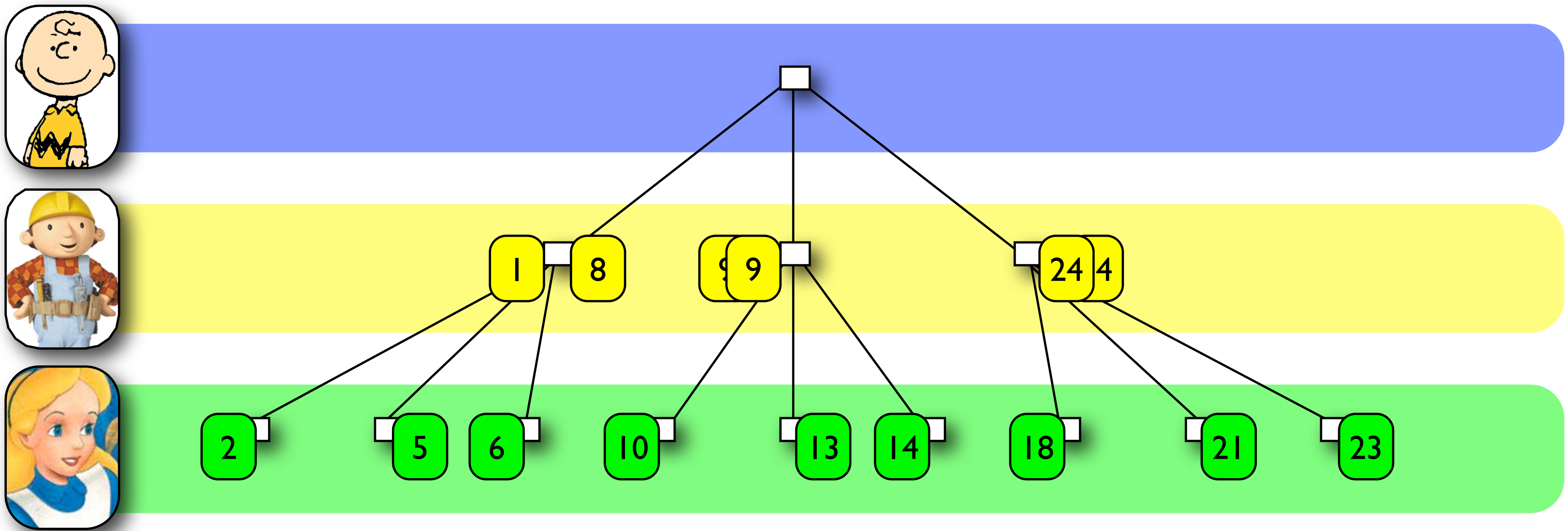

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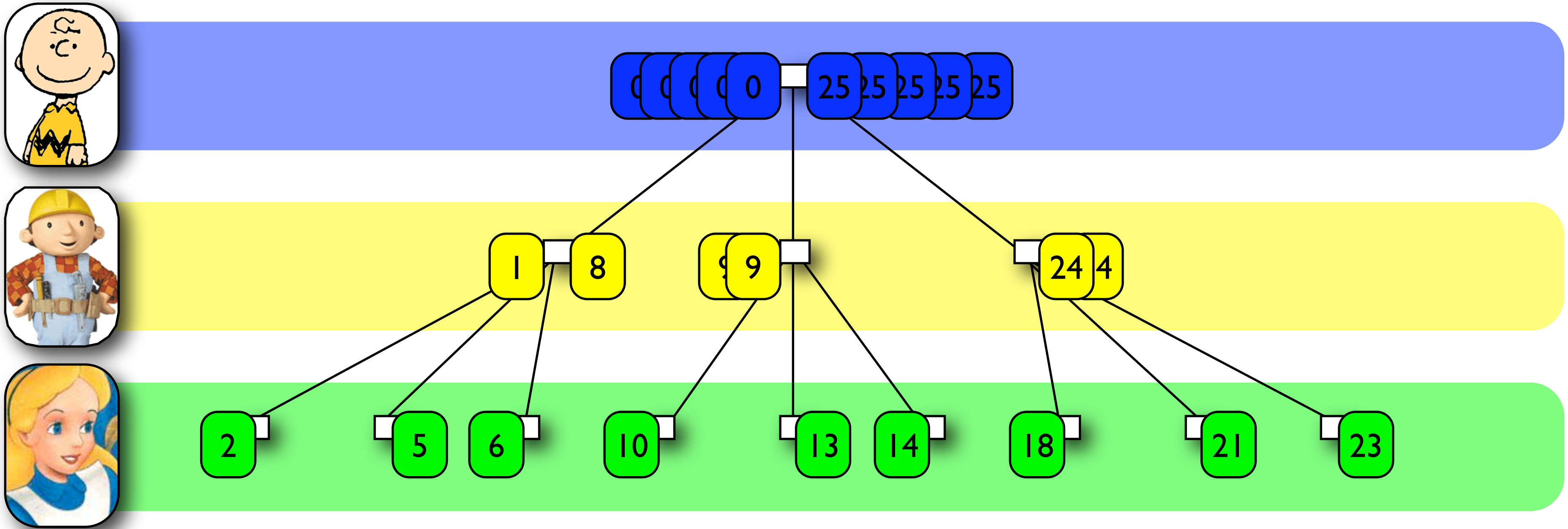

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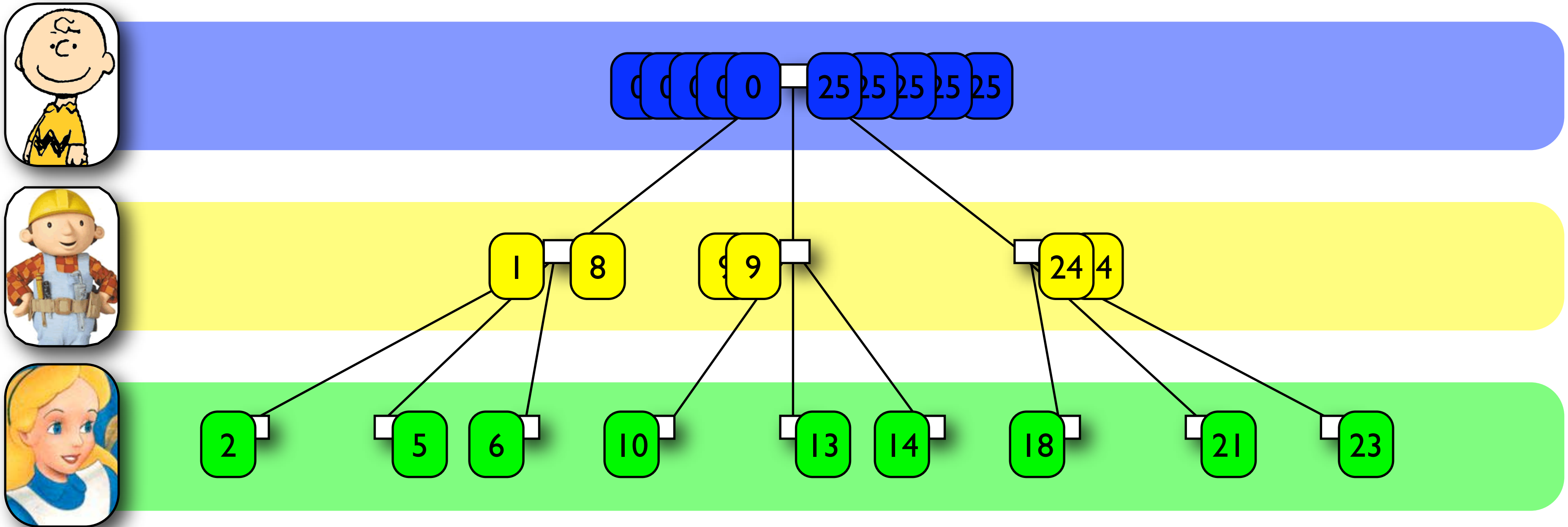

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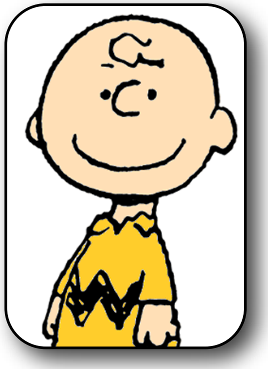
- Relationship between *t* and # copies determines bound.

Simulating Random-Partition Protocol...



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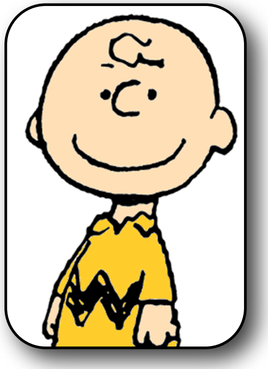

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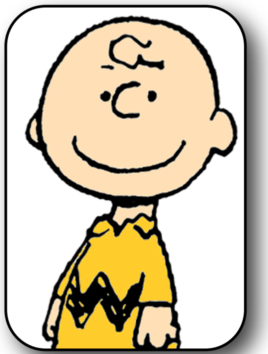

α α α β β β β β

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  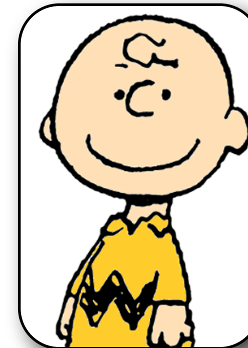

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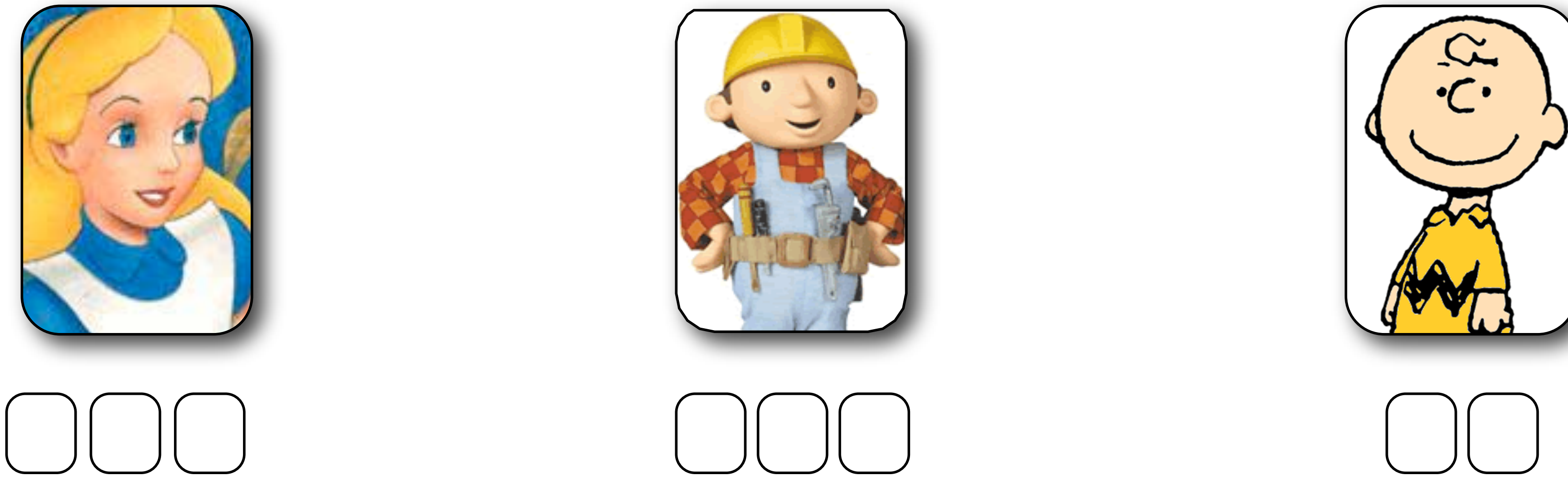

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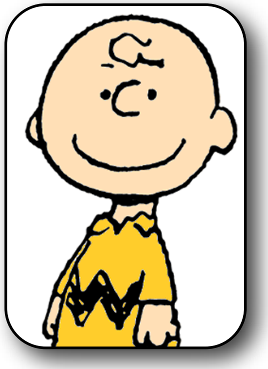
α β α



α β

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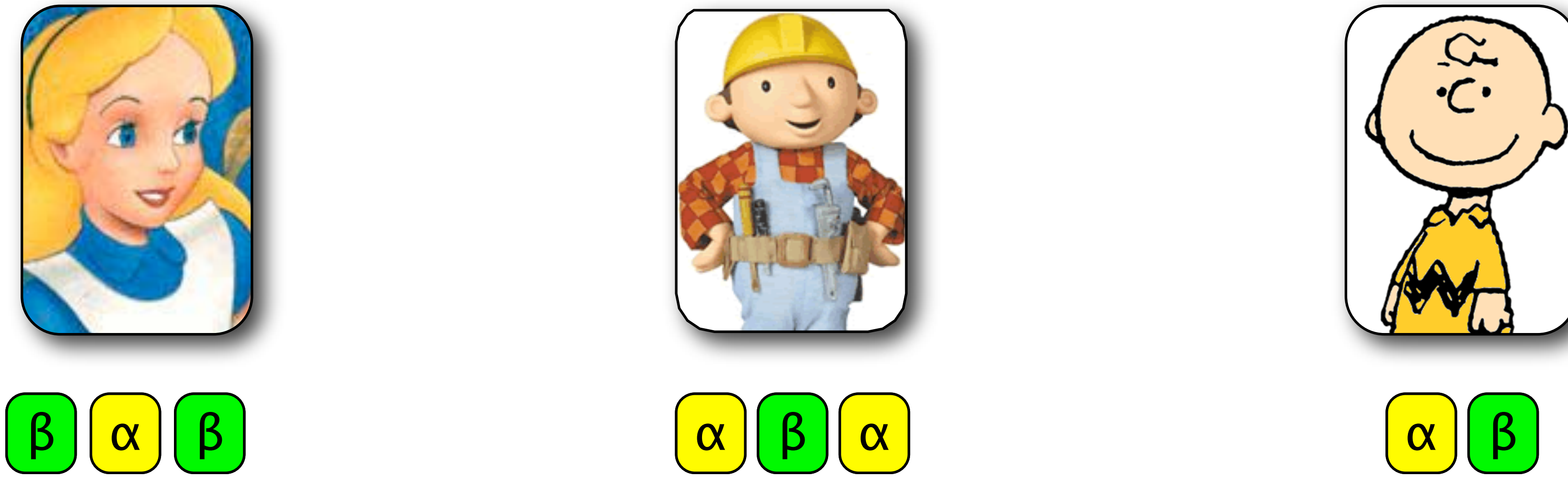

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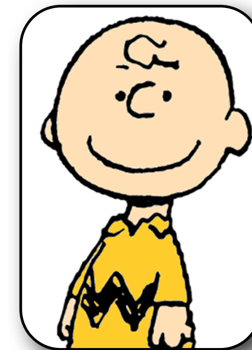

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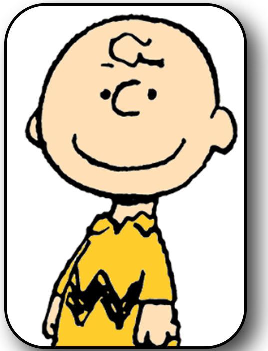

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- *Thm:* Partition looks random if total number of tokens is greater than (max bias)^2. Hence, $m = \exp(2^p \lg t)$.

Summary

Introduced notion of Robust Lower Bounds

Tight communication bounds for *disjointness*, *indexing*, *gap-hamming*, and improved *selection* bound.

Data streams bounds including *frequency moments*, *connectivity*, *entropy*, F_0, *quantile estimation*, ...

Many open problems... *Thanks!*

“Step 2” Simulation...

- Need protocol for fixed-partition AND_t using protocol for random-partition $\mathrm{DISJ}_{n.t}$.
- *Simulate Π (for disjointness) to solve* AND_t*:*

  a) Using public coin, create matrix X with j^{th} column X^j

  b) Using public coin, partition X between p virtual players

  c) Run Π: player i simulates virtual player with i^{th} bit of X^j

  (*Give up if a virtual player receives two bits from* X^j)

- *Failure probability:* δ+Birthday(t,p).

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$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \left\{ \begin{pmatrix} & 0 & & \\ & & 0 & 0 \\ & & 1 & \end{pmatrix}, \begin{pmatrix} 0 & & 0 & 1 \\ 0 & 1 & & \\ & & & \end{pmatrix}, \begin{pmatrix} & & & \\ & & & \\ 0 & 0 & & 0 \end{pmatrix} \right\}$$

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