Distribution of the number of distinct sites visited by random walks in disordered lattices

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The distributions of the number of distinct sites S_n visited by random walks of n steps in the infinite cluster of two-dimensional lattices at the percolation threshold are studied. Different lattice sizes, different origins of the walks, and different realizations of the disorder are investigated by Monte Carlo simulations. The distribution of the mean values of $\langle S_n \rangle$ appears to have self-averaging features. The probability distribution of the normalized values of $\langle S_n \rangle$ is investigated with respect to its multifractal behavior. The distributions of the probabilities $p(S_n)$ for fixed S_n are presented and analyzed. These distributions are wide and their moments show behavior that cannot be characterized by multifractal scaling exponents.

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I. INTRODUCTION

In this paper we investigate some properties of the distributions of the number of sites that are visited by random walks in lattices at the percolation threshold. The infinite cluster of accessible sites in lattices with randomly blocked sites is a fractal at the percolation threshold. Consequently, the mean number of distinct sites visited by random walks shows a power-law dependence on the step number [1]. In particular we ask whether the *distributions* of the number of distinct sites visited exhibit multifractal features. In the multifractal analysis the structure of the probability distributions themselves are of interest [2], for instance, their dependence on length scales, or the magnitude of the fluctuations between different realizations of the disorder. With regard to the number S of visited sites, we can investigate the structure of the distribution for different lattice sizes, for different starting sites of the random walks in a particular realization of the infinite cluster, and for different realizations of the infinite cluster. These points are discussed in this paper.

Multifractal properties of random walks in percolation lattices were already investigated by Bunde, Havlin, and Roman [3] who studied the probability distribution of the displacements R of particles. Especially they considered the distributions of the probability distribution, for fixed distance R and step number n. They were able to predict multifractal properties of these distributions by comparing them with simpler models, and they confirmed the ensuing scaling predictions by numerical methods. We wish to point out why we think it is worthwhile to also undertake a study of the multifractal features of the distributions of S_n. The number S of distinct sites visited is a quantity that is different from the displacements of a random walk. It is related to the first-passage times of the walk to the lattice points. The distance R of a walk belongs to the class of properties that are associated with the mean-square displacement. The mean number $\langle S_n \rangle$ of distinct sites visited, and the mean-square displacement $\langle R^2 \rangle$ have quite different properties in the infinite percolation cluster. While the power-law behavior of $\langle R^2 \rangle$ is described by the random-walk exponent d_W [4], the behavior of $\langle S_n \rangle$ with the number of steps n is determined by the spectral dimension d_S [5]. The difficulty of studying the distribution of S_n is that no useful simple models are known for predicting the scaling behavior of the distributions, and that the exact enumeration technique is not applicable.

In addition to the analysis of the probability distributions of S_n, which is analogous to the analysis of Bunde *et al.* [3] we also investigate the properties of the distributions of S_n by box-counting methods, as considered by Halsey *et al.* [6]. Such methods are quite natural to study, e.g., strange sets in turbulence [7] and in chaotic systems [8]. One application of the box-counting method to a first-passage time problem was made recently by Murthy *et al.* [9]. We extend this method to the study of the distribution of the values of S_n in different realizations of the random walks.

By using two different methods of investigating possible multifractal properties of a particular quantity, we also want to contribute to a better understanding of the concept of multifractality, when applied to disordered condensed-matter systems. Systems with quenched disorder are quite different from some of the model systems that are studied in this context, and the interpretation of multifractality and its consequences for observable properties are still not well understood.

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II. THE MODEL

Two-dimensional simple-square lattices are considered, in which some sites are randomly blocked and thus inaccessible. The concentration of open, accessible, sites is designated by p. This model represents the well-known percolation problem [10]. For small values of p and large lattices the clusters of open sites are not connected, while for large p there exists an "infinite cluster" of open sites which connects sites at different boundaries of arbitrarily large lattices. A well defined critical point separates the two regimes; for the simple-square lattice the critical point is located at $p_c = 0.5931$. The critical point is of special significance, as physical quantities show scaling behavior at or near this point, and the infinite cluster has fractal properties.

To investigate random-walk properties in this model, a particle is placed at random on an open site of the lattice and it makes transitions to open neighbor sites. We consider discrete random walks where the number of steps n is counted. We adopt the "blind-ant" convention of counting also attempted transitions [10]. If the particle attempts to make a transition to a blocked site, it is not allowed to do so, but the number of steps is increased by one.

The prescription of placing a particle initially on a open site of the lattice at the percolation threshold has to be explained in detail. The critical behavior of the mean-square displacement $\langle R_n^2 \rangle$ and of the mean number of distinct sites visited, $\langle S_n \rangle$, is different, depending on whether an average is done over walks of the particle in the infinite cluster only, or whether the average includes all clusters of open sites [4,5]. In the first case, the starting point of the particle is always chosen in the infinite cluster, in the second case the starting point is an arbitrary open site. We study the first case, i.e., we consider only walks in the infinite cluster of the percolating lattice.

Lattices of several different sizes were used, ranging from 100×100 up to 750×750. Sites were selected randomly with probability $1 - p$ and designated as blocked sites. At the percolation threshold p_c, the "infinite cluster" of open sites was constructed by the cluster multiple-labeling technique [11]. After preparing the disordered lattice, a particle is placed randomly on an open site of the infinite cluster, and allowed to perform a random walk. The number of visited sites S is monitored as function of the step number n.

III. DETERMINATION OF THE DISTRIBUTION OF S_n

In this paper we are concerned with the different behavior of random-walk quantities in the different realizations of the disorder. These may be implemented by choosing different starting points of the random walks, or by creating different realizations of the infinite cluster at the percolation threshold. We first consider random walks that start exactly at the same point of the percolation cluster, for one realization of the disorder. We perform N_{RW} random walks with a fixed number n of steps that start at the same origin and estimate the distribution $p(S_n)$ of the number of distinct sites visited. We also determine the mean value $\langle S_n \rangle$ of distinct sites visited for this realization of the disorder. We then choose another origin and determine a second distribution $p(S_n)$ and the mean value $\langle S_n \rangle$ by performing again N_{RW} random walks that start exactly at the second origin. We repeat the procedure of determining different distributions $p(S_n)$ and values of $\langle S_n \rangle$ for N_r selections of a different origin. The origins comprise all sites of the infinite cluster.

In addition to the determination of $p(S_n)$ for different origins, in a given realization of the infinite cluster, we also investigated different realizations of the infinite cluster at the percolation threshold. For the lattices of size 250×250 we repeated the analysis described above for another realization of the infinite cluster, again with N_r selections of a different origins. In total, $\mathcal{N} = 12$ different realizations of the disorder were created and investigated. The number of random walks performed from one origin was typically $N_{\mathrm{RW}} = 10\,000$, the step number was always fixed at $n = 1000$, and $N_r = 10\,000 - 22\,000$ realizations of different origins for each of the $\mathcal{N} = 12$ realizations of the disorder were taken. As a result we have then $N_r \times \mathcal{N}$ different distributions $p_{i,\kappa}(S_n)$ $(i = 1, \ldots, N_r, \kappa = 1, \ldots, \mathcal{N})$ and $N_r \times \mathcal{N}$ values of $\langle S_n \rangle$. The further analysis of these quantities is described in the following section.

To examine the dependence on the lattice size distributions of the probability distributions of the $\langle S_n \rangle$ values we also created one "infinite" cluster in a lattice of 750×750 sites. The infinite cluster had 180 605 sites and we used all these sites as starting sites in the procedure described above. In this way we obtained an additional set of $N_r = 180\,605$ distributions $p_i(S_n)$ and values of $\langle S_n \rangle$.

It is not evident that the use of different origins of the random walks in one realization of the disorder is equivalent to the procedure of creating many different realizations of the disorder for the random walks. To address this question we also created 10 000 different realizations of the infinite cluster in lattices of sizes 250×250 and performed $N_{\mathrm{RW}} = 10\,000$ random walks for one origin in each realization.

In the remainder of this section we consider the distribution of the mean values $\langle S_n \rangle$ of distinct sites visited by random walks of n steps that arise from the different selections of the origins in fixed realizations of the disorder, and from the 10 000 different realizations of the disorder, with only one origin of the walks. All $\langle S_n \rangle$ values that were obtained by the procedure described above are rounded to integers. We search for the smallest value of $\langle S \rangle$ in this set and note how often it occurs. We continue by increasing $\langle S_n \rangle$ by unity and noting its number of occurrence. This procedure is repeated up to the maximal value of $\langle S_n \rangle$. In this way we obtain a histogram of the normalized distribution W of the $\langle S_n \rangle$ values. Three such distributions are shown in Fig. 1, namely, for the infinite cluster in the lattice with 750×750 sites, for the combined histogram of $\langle S_n \rangle$ values for all ($\mathcal{N} = 12$) infinite clusters of the lattices with 250×250 sites, and also

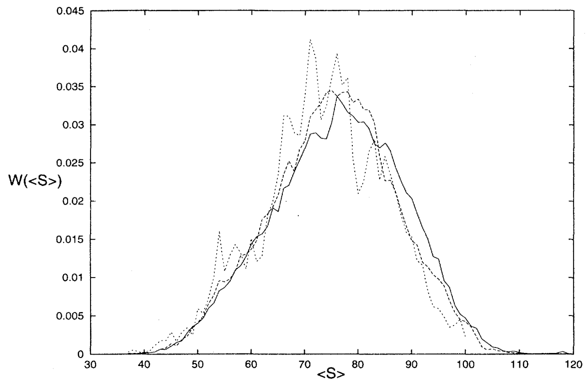

FIG. 1. Distribution W of $\langle S_n\rangle$, the mean number of distinct sites visited by random walks in the infinite cluster of a percolating lattice. The solid line represents the distribution for a cluster of size 750×750, where $N_r = 180\,605$, the dotted line represents the distribution for a cluster of size 250×250 with $N_r = 14\,550$, while the dashed line represents the average of 12 different clusters of size 250×250.

for one infinite cluster in a lattice with 250×250 sites. The fact that we have a broad distribution demonstrates that quite different values of $\langle S_n\rangle$ appear for different selections of the origin.

All three distributions show fluctuations in their shape. The distribution $W(\langle S_n\rangle)$ shows less fluctuations when it is determined in the largest lattice or as an average over the 12 smaller lattices, than in the case of one smaller lattice. The fluctuations of the shape of $W(\langle S_n\rangle)$ for the largest lattice are of the same order as for the average over the 12 smaller lattices. Both samples contain approximately the same number of origins. This means that the fluctuations are averaged out by choosing larger lattices and correspondingly more different origins in the infinite cluster.

In Fig. 2 we give the distribution $W(\langle S_n\rangle)$ that results

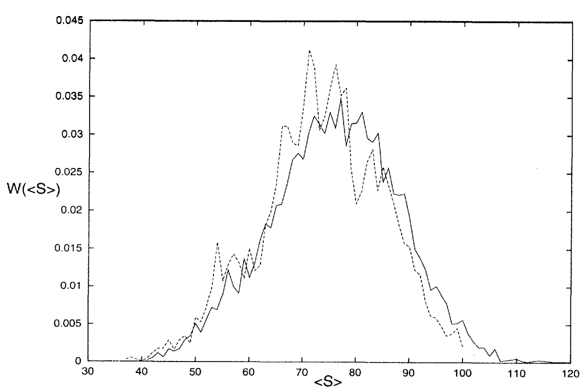

FIG. 2. Comparison of the distributions W of $\langle S_n\rangle$, the mean number of distinct sites visited by random walks for $\mathcal{N} = 10\,000$ different realizations of the percolating cluster in lattices of size 250×250 (solid line), and for one realization of the cluster of the same size (dashed line), where $N_r = 14\,550$ different realizations of the origin have been considered.

from the creation of $\mathcal{N} = 10\,000$ different realizations of the infinite cluster, with one origin in each of the realizations. For comparison, the distribution $W(\langle S_n\rangle)$ for one infinite cluster in a 250×250 lattice and for all its sites taken as origins, is also given. We observe that both distributions show the same magnitude of fluctuations. Hence we can state that the choice of different origins in one realization of the infinite cluster leads qualitatively to the same behavior of $W(\langle S_n\rangle)$ as the implementation of different realizations of the disorder. Since the fluctuations are averaged out by choosing larger lattices, we can say that the distribution $W(\langle S_n\rangle)$ appears to be a self-averaging quantity.

IV. MULTIFRACTAL ANALYSIS

A. Distribution of mean value $\langle S_n\rangle$

In principle, one can use the distribution $W(\langle S_n\rangle)$ of the previous section for further analysis, for instance, with regard to the behavior of its moments. In this subsection we will proceed with the analysis of the distribution of the $\langle S_n\rangle$ values in a different way, which is more in vein with other analyses of multifractality [6,9]. We note that the maximal possible number of visited sites is $S_{\max} = n$ and the minimal number is $S_{\min} = 1$, although these values are never realized in practice. We define the normalized quantity $\tilde{s}$ by

$$\tilde{s} = \frac{\langle S_n\rangle - S_{\min}}{S_{\max} - S_{\min}} . \qquad (1)$$

We now make a box subdivision of the possible values of $0 < \tilde{s} < 1$ into 2^ν boxes and put the $\tilde{s}$ values obtained into the boxes. We define

$$\mu_i = \frac{N_i}{N_r}, \qquad (2)$$

where N_i is the number of $\tilde{s}$ values that fall into box number i, and N_r is the number of different origins of random walks that we use in our calculations.

Figures 3(a)–(d) show the resulting histograms of the distributions of the $\tilde{s}$ values, for different values of ν. One recognizes strong fluctuations, for the different choices of the ν values. Qualitatively, the fluctuations for $\nu = 10$ have the same magnitude as the fluctuations in the distribution $W(\langle S_n\rangle)$ that is shown in Fig. 1. For the ν values that are used in Fig. 3 the individual probabilities μ_i take on quite different values. If ν is increased, the subdivision becomes finer and finer, and eventually the situation is reached where the unnormalized N_i in Eq. (2) are either zero or one. This is certainly the case for ν larger than ≈ 24.

To get information on the structure of the distribution of the μ_i we calculate the partition function from the moments

$$Z_q(l) = \sum_i \mu_i^q , \qquad (3)$$

where $l = 2^{-\nu}$, and investigate whether

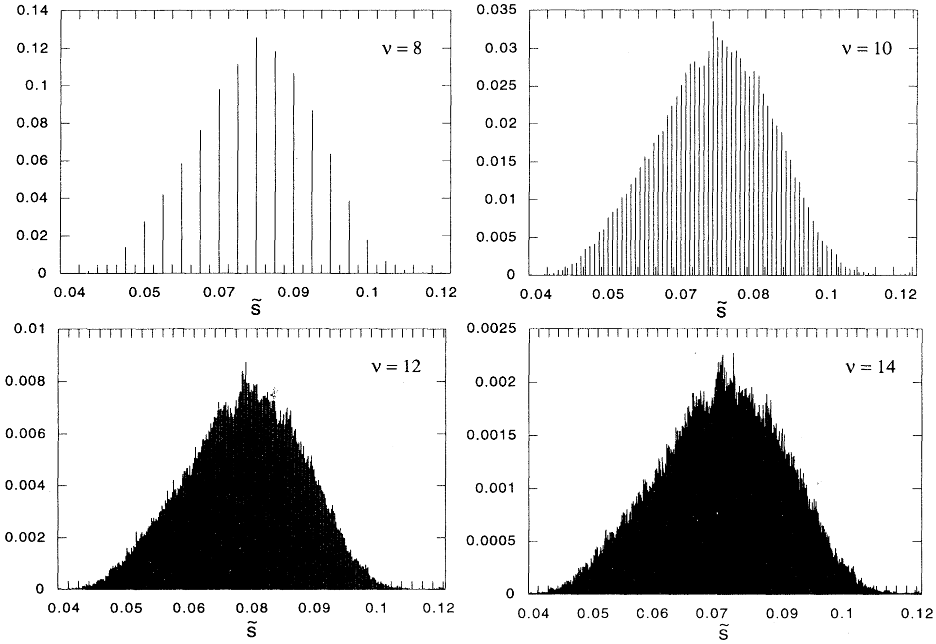

FIG. 3. Histogram of the probabilities of occurrences of $\tilde{s}$ values for four box subdivisions, as shown in the figures. The data were obtained from the lattice of size 750×750.

$$Z_q(l) \sim l^{\tau(q)} \sim 2^{-\nu\tau(q)} \tag{4}$$

holds. For this purpose in Fig. 4 we have plotted $\ln Z_q(l)$ versus l, for various q values. The symbols are the calculated Z values, while the continuous lines are the fits to the data. The dotted lines are the slopes $\tau_h(q) = q - 1$, where the index h gives the homogeneous case (nonmultifractal) behavior. For all positive q one can easily see a coincidence of the data with this latter behavior. For negative q's the curves exhibit a break in the slope at $\nu = 12$. Figure 5 gives a plot of $\tau(q)$ vs q. Here the dots are the data resulting from the slopes that were fitted in Fig. 4, with a $\Delta q = 0.1$ interval. The solid line is a straight line with a slope of $\tau_h = q - 1$.

We observe a curved behavior of $\tau(q)$ versus q which would generally indicate multifractal behavior of the underlying measure. The $\tau(q)$ curve that is obtained from the smaller ν's for negative q is convex, as in other examples of multifractal distributions [3,6,9,12]. However, the curvature is not very pronounced, and the slope is near 1

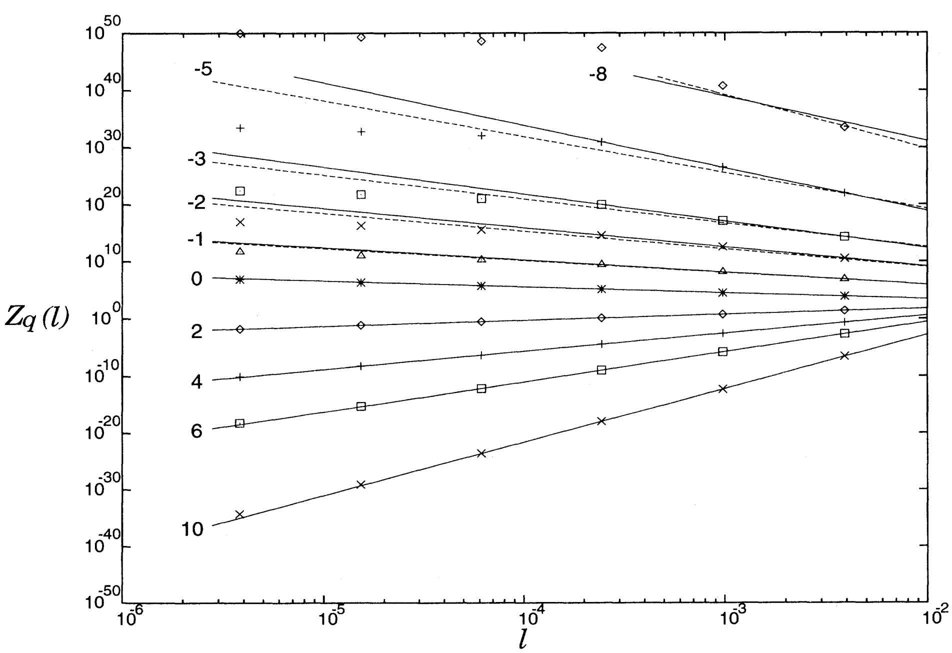

FIG. 4. Plot of $Z_q(l)$ versus l in a log-log representation, for different q values (q values are shown at the left of the curves). The symbols are the calculated Z values, while the continuous lines are the fits to the data. The dotted lines are the homogeneous slopes $\tau_h(q) = q - 1$. Note the crossover in the slopes of the negative moments, which occurs at about $\nu = 12$.

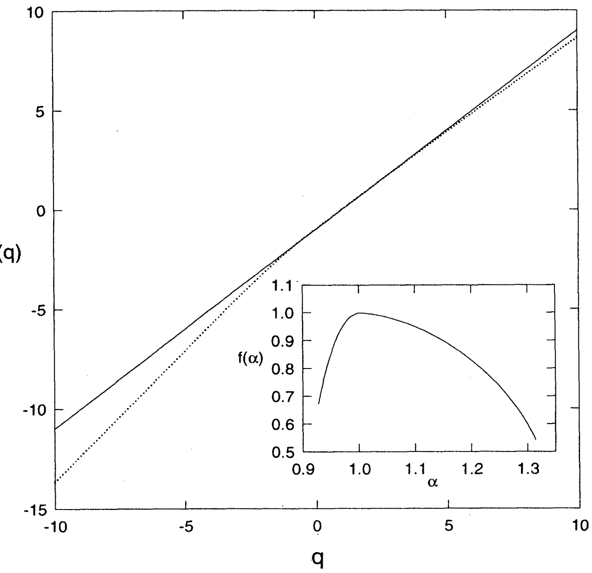

FIG. 5. Plot of the mass scaling exponents $\tau(q)$ vs q that follow from the curves of Fig. 4. The dots represent the slopes of the curves of Fig. 4 for small ν values, while the continuous line is the straight line $\tau_h(q) = q - 1$. The inset is a plot of the singularity density that results from the mass scaling exponents.

for positive q. A slope of exactly 1 corresponds to uniform measures and constant-gap scaling. It may well be that $\tau(q)$ becomes completely straight for infinitely large lattices. The $\tau(q)$ curve that is determined from the larger ν's for negative q is concave, contrary to other cases of multifractality. We cannot offer arguments whether such a behavior is reasonable, and what would be the limiting value of $\tau(q)$.

Note that $\tau(1) = 0$, because of the normalization of the μ_i. For $q = 0$, the moments $Z(0) = N_k$ where N_k is the number of boxes with nonzero entries in the corresponding subdivision. The resulting value $\tau(0)$ is the negative of the fractal dimension of the set [6], i.e., $D(0) = -\tau(0)$. We observe $D(0) = 0.96$. This value is close to unity, indicating that there are only a few empty boxes in the subdivisions between the bounds of the distribution.

Another quantity of interest in the multifractal analysis is the singularity density $f(\alpha)$. It is obtained from the scaling exponents by a Legendre transformation

$$f(\alpha) = \alpha q - \tau(q), \qquad \alpha = \frac{d\tau(q)}{dq} . \tag{5}$$

To obtain the singularity density $f(\alpha)$ we have differentiated the $\tau(q)$ curves of Fig. 5 numerically using intervals $\Delta q = 0.1$. The results for the singularity density are shown in the inset of Fig. 5, and do not show the usual behavior that is characteristic of multifractal measures. The details of this curve can be understood from the behavior of $\tau(q)$, e.g., the sections that are almost vertical originate from the $\tau(q)$ for positive q. In summary, we must leave the question open whether the distribution of the normalized values of $\tilde{s}$ is multifractal or not.

B. Distribution of probabilities of visiting certain numbers of sites

We now investigate in detail how the probability that a certain number of distinct sites is visited depends on the choice of the origins of the walks, in given realizations of the disorder, and on the different realizations of the infinite cluster at the critical point. It was described in Sec. III how the different distributions $p_{i,\kappa}(S_n)$ are estimated by numerical simulations. We first consider the case where different origins were chosen, for a given realization of the infinite cluster. Since the origins of the random walks may have quite different local vicinities, we expect widely different distributions of the numbers of distinct sites visited, for different origins.

Figure 6 shows four different distributions (taken at random) of the number of distinct sites visited, for random walks starting at four different origins. The distributions $P(S_n)$ are not yet normalized. The distributions shown in Fig. 6 are very different from each other; evidently this is a consequence of the form of the infinite cluster at the percolation threshold. If the distributions $P(S_n)$ are determined in a lattice away from the percolation threshold, they should not so strongly differ from each other. We have convinced ourselves that this is indeed the case, and a figure for $p = 0.75$ is given in Ref. [13], in which the four distributions of Fig. 6 practically collapse.

Next we consider the set of different realizations of $p(S_n)$ for N_r different origins in a fixed realization of the infinite cluster. Let us consider a specific value of S_n, for instance, $S_n = 100$. The multifractal analysis is concerned with the specific properties of the probabilities themselves. Here the question is the distribution of these probabilities $p(S_n)$, for fixed S_n, for different selections of the origins. A related problem is the structure of the distribution of the individual probabilities $p(S_n)$, taken at the mean values $S_n = \langle S \rangle_\alpha$ for the different realizations

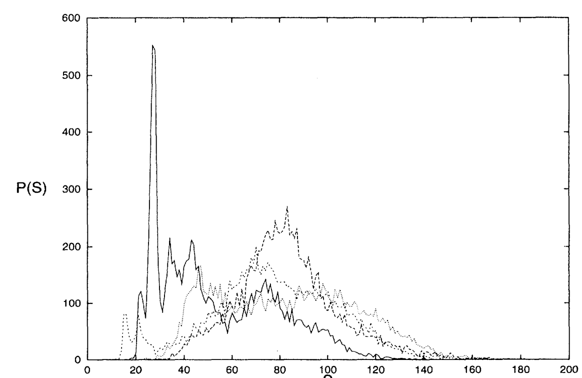

FIG. 6. Plot of four distributions $P(S)$ of the number S of sites visited by random walks starting at four different points of the percolating cluster at the critical threshold. Each distribution is made from 10 000 realizations of walks of $n = 1000$ steps which start at a fixed origin. The lattice size here is 250 $\times$ 250.

of the origins.

To visualize the distribution of $p(S_n)$ for fixed S_n we consider the set of the unnormalized distributions $P(S_n)$ and represent the number of occurrences of different values of P for fixed S_n in a histogram. For simplicity we call the distribution of the different values of $P(S_n)$ a "hyperdistribution," $\Pi(P(S_n))$. Four representative hyperdistributions for four specific values of S_n are given in Fig. 7. We find the overall average of S_n for $n = 1000$ to be about $\{\langle S_n\rangle\} \approx 76$, and the brackets designate the random-walk average for fixed origin, while the braces signify the disorder average. We notice that for S_n values that are considerably smaller or larger than this average value the hyperdistributions are skewed to small values of P. For S_n close to the overall average the distributions are centered around a value which may be considered as typical.

In the multifractal analysis of these distributions, the moments of the set of the probabilities are determined,

$$m_q = \frac{1}{N_r}\sum_{i=1}^{N_r} p_i^q(S_n)\,, \tag{6}$$

where the sum runs over the different realizations of the disorder. The moments can also be derived from the hyperdistribution $\Pi(P(S_n))$ after suitable normalization, but we prefer to work directly with the definition (6). The exponent q may be taken positive, or negative with the proviso that zero values of $p(S_n)$ are omitted for $q < 0$. The probabilities $p(S_n)$ are normalized with respect to summation over S_n,

$$p(S_n) = \frac{P(S_n)}{N_{\rm RW}}, \qquad \sum_{S_n} p(S_n) = 1\,. \tag{7}$$

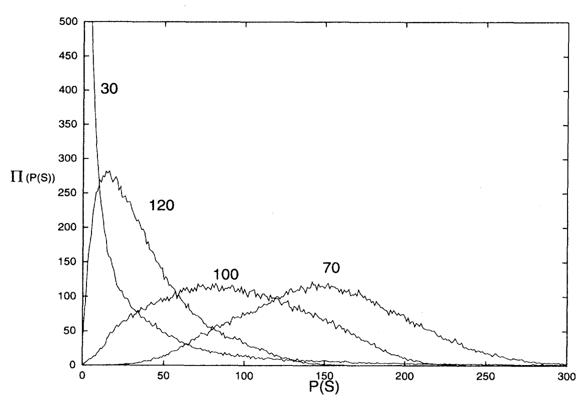

FIG. 7. Plot of the distributions Π of the probabilities $P(S_n)$ when all the origins of 12 different realizations of the percolation cluster are taken, resulting to a total of almost 180 000 origins. Four values of S_n are considered, as marked. The x axis in this figure corresponds to the y axis of Fig. 6. The y axis in this figure now shows the distribution of the P values for about 180 000 different origins for the four specific S_n values. For each origin $P(S_n)$ was determined with $n = 1000$ steps and 10 000 realizations of the walks, as the data in Fig. 6.

Consequently, in our analysis the moment m_1 is not normalized to unity. Instead, it represents an estimate of the disorder average of $p(S_n)$ and it will serve as a reference quantity.

Since this quantity plays an important role in the subsequent analysis, we will show its dependence on the value of S_n that is selected for the corresponding hyperdistribution. Figure 8 presents the first moment, designated by $\{p\}$, as a function of the considered values of S_n. The curve shows very small fluctuations to the left of the maximum, but it is completely smooth on the right side. The maximum corresponds to the most probable value of S_n. Similar curves can be produced for the other moments. The positive moments show increasing fluctuations with q to the left of the maximum. The curves for the negative moments have two maxima, as it is expected.

The mass scaling exponents $\tau(q)$ can be defined as

$$m_q = (m_1)^{\tau(q)}\,. \tag{8}$$

From the definition of the moments (6) follows obviously $\tau(1) = 1$. If the probabilities $p(S_n)$, are all different from zero in the different realizations of the origins, then $m_0 = 1$ and consequently $\tau(0) = 0$. It should be noted that in this characterization of the multifractality of the set of different realizations the system size is not a relevant parameter, similar to the study of Ref. [3]. The scaling parameter of this approach is the first moment m_1, as Eq. (8) shows.

The application of Eq. (8) requires that the moments m_q scale indeed in this way under variations of m_1. This property must first be examined for the probability distributions under study. Figure 9 gives a plot of m_q versus $m_1 \equiv \{p\}$, in a double-logarithmic representation, for various values of q. There appear two branches of the curves m_q versus m_1, corresponding to two different S_n values. It is evident that no straight lines are present (except for $q = 1$, which is the trivial case), only some

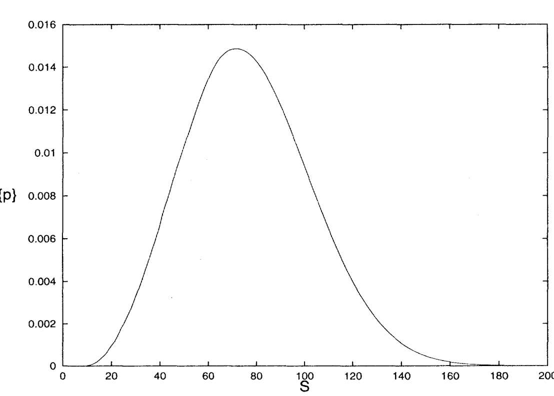

FIG. 8. Plot of the distribution of the first moment of p, i.e., the average value $\{p\} \equiv m_1$, with regard to the number of sites visited, for a walk of $n = 1000$ steps, for 12 different realizations of the percolation cluster.

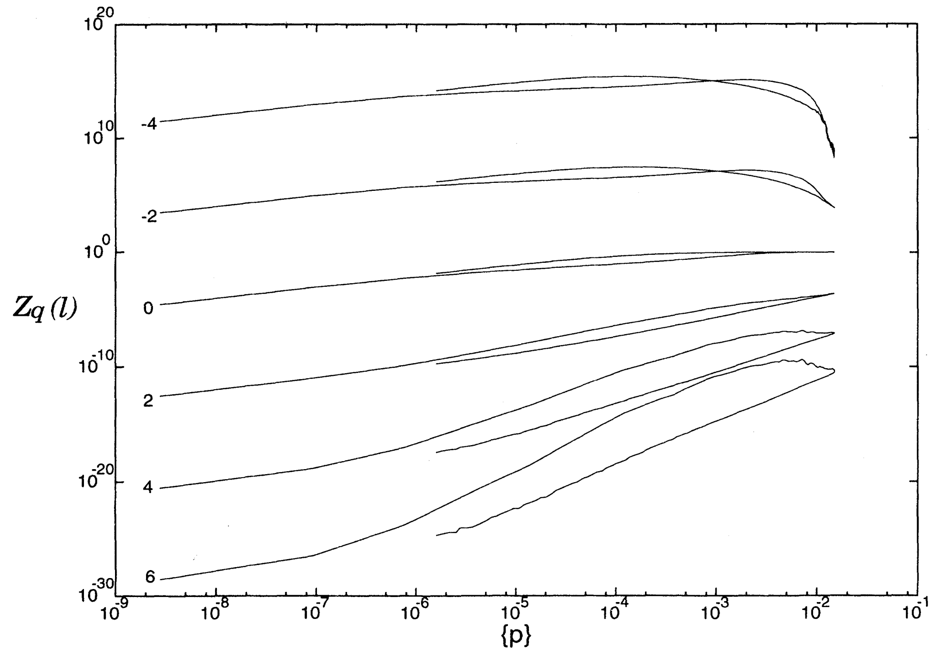

FIG. 9. Plot of the partition function $Z_q(l)$ as a function of the first moment $\{p\} \equiv m_1$, for different values of q (q values are shown at the left of the curves).

sections of the curves are straight. Evidently a determination of $\tau(q)$ from these curves would be ambiguous. We must conclude that the scaling property expressed by Eq. (8) does not hold for the set of probabilities p_i, and this set, or the corresponding hyperdistribution $\Pi(p(S_n))$ is not a multifractal. Apparently it is a set that cannot be characterized by multifractal analysis; it seems to be a more complicated quantity.

V. SUMMARY AND DISCUSSION

In this paper we studied distributions $p(S_n)$ of the number S_n of distinct sites visited by random walks of n steps in the "infinite cluster" of two-dimensional lattices at the percolation threshold. We tried to look into several aspects of the disorder present in our system, and thus we used different lattice sites, different origins of the random walks within the infinite cluster, and different lattice clusters.

In the first portion of the paper we directed our attention to the mean values $\langle S_n \rangle$ of distinct sites visited in the particular realizations of origins or the disorder. The distribution $W(\langle S_n \rangle)$ of the mean values is broad and shows fluctuations of its shape that depend on the size of the lattices used. There is no qualitative difference between the use of different origins in one realization, and the use of many different realizations of the disorder. Since the fluctuations are diminished with increasing lattice size we surmise that $W(\langle S_n \rangle)$ is a self-averaging quantity.

We then analyzed the distributions of the normalized values of $\langle S_n \rangle$ over boxes of varying lengths $l = 2^{-\nu}$. The moments of the distributions scale with the length of the boxes for positive powers, but this is not the case for negative powers. Hence the multifractality of the set of the $\langle S_n \rangle$ values is not yet firmly established. It may well be that one finds constant-gap scaling for infinitely large lattices.

In the second portion of the work we discussed the set of the probability distributions $p(S_n)$ for fixed numbers of distinct sites visited, in different realizations of the origins of the walks, corresponding to different realizations of the disorder. The individual probability distributions $p(S_n)$ show widely different behavior. For fixed S_n, the distribution of the probabilities $p(S_n)$ was studied. The dependence of its first moment on S_n is well characterized. We find that the higher moments of the distribution of the probabilities do not scale in a unique way with the first moment of this distribution. Hence we cannot obtain a useful mass scaling exponent $\tau(q)$. We gained the impression that the distribution of the probabilities $p(S_n)$ is a more complicated quantity than can be described by multifractal analysis.

In summary, we gained considerable insight in the probability distributions of particular random-walk quantities, S_n or $\langle S_n \rangle$, and in the distribution of the probabilities over different realizations of the disorder. These distributions are broad, however, and they are not easily amenable to multifractal analysis. We conclude that multifractal behavior is restricted to special circumstances and not generally associated with quenched disorder.

ACKNOWLEDGMENTS

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